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ON SS-QUASINORMAL AND WEAKLY S-PERMUTABLE SUBGROUPS OF FINITE GROUPS

Changwen Li

Abstract. A subgroup H of a group G is called ss-quasinormal in G if there is a subgroup B of G such that $G = HB$ and H permutes with every Sylow subgroup of B; H is called weakly s-permutable in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are s-permutable in G. We fix in every non-cyclic Sylow subgroup P of G some subgroup D satisfying $1 < |D| < |P|$ and study the structure of G under the assumption that every subgroup H of P with $|H| = |D|$ is either ss-quasinormal or weakly s-permutable in G. Some recent results are generalized and unified.

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1. Introduction

Throughout this paper, all groups are finite. Most of the notions are standard and can be found in [1]. G denotes always a group, $|G|$ is the order of G, $\pi(G)$ denotes the set of all primes dividing $|G|$, G_p is a Sylow p-subgroup of G for some $p \in \pi(G)$, $O_p(G)$ is the maximal normal p-subgroup of G and $O_{p'}(G) =< G_q \mid q \in \pi(G), p \neq q >$.

Recall that a class $\mathcal{F}$ of groups is called a formation if $\mathcal{F}$ contains all homomorphic images of a group in group in $\mathcal{F}$, and if G/M and G/N are in $\mathcal{F}$, then $G/(M \cap N)$ is in $\mathcal{F}$. A formation $\mathcal{F}$ is said to be saturated if $G/\Phi(G) \in \mathcal{F}$ implies that $G \in \mathcal{F}$. In this paper, $\mathcal{U}$ will denote the class of all supersolvable groups. Since a group G is supersolvable if and only if $G/\Phi(G)$ is supersolvable [1, p. 713, Satz 8.6], it follows that $\mathcal{U}$ is saturated.

Two subgroups H and K of G are said to be permutable if $HK = KH$. A subgroup H of G is said to be s-quasinormal (or s-permutable, π-quasinormal) in G

if H permutes with every Sylow subgroup of G [2]. This concept was introduced by O.H.Kegel in 1962 and was investigated by many authors. Recently, s-quasinormal subgroups are extended to all kinds of forms. Form example, Li, etc. [3], introduced the following concept of ss-quasinormality:

Definition 1. *A subgroup H of G is called ss-quasinormal in G if there is a subgroup B of G such that $G = HB$ and H permutes with every Sylow subgroup of B.*
and A.N.Skiba [6] gave the following concept of weakly s-permutability:

Definition 2. *A subgroup H of G is called weakly s-permutable in G if there is a subnormal subgroup T of G such that $G = HT$ and $H \cap T \leq H_{sG}$, where H_{sG} is the subgroup of H generated by all those subgroups of H which are s-permutable in G.*
It is easy to find groups with ss-quasinormal subgroups which are not weakly s-permutable. Conversely, there are also groups with weakly s-permutable subgroups which are not ss-quasinormal.

Example 1. Let $G = A_5$, the alternative group of degree 5. Then A_4 is ss-quasinormal in G, but not weakly s-permutable in G.

Example 2. Let $G = S_4$, the symmetric group of degree 4. Take $H =< (34) >$. Then H is weakly s-permutable in G, but not ss-quasinormal in G.
The structure of a group G under the assumption that some minimal or maximal subgroups of the Sylow subgroups are well situated in G has been investigated by many authors in the literature, such as in [3, 4, 5, 11, 12, 13], etc. In the nice paper [6], Skiba gave a unified result as follows.

Theorem A *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is weakly s-permutable in G, where $F^*(E)$ is the generalized Fitting subgroup of E. Then $G \in \mathcal{F}$.*
In this paper, the aim of this article is to extend Theorem A as follows and unify some earlier results using ss-quasinormal and weakly s-permutable subgroups.

Theorem B *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersolvable groups and G a group with E as a normal subgroup of G such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a nonabelian 2-group and $|P : D| > 2$) is either ss-quasinormal or weakly s-permutable in G, where $F^*(E)$ is the generalized Fitting subgroup of E. Then $G \in \mathcal{F}$.*

2. Preliminaries

Lemma 1. *Let H be an ss-quasinormal subgroup of a group G.*

(*a*) *If $H \leq L \leq G$, then H is ss-quasinormal in L.*

(*b*) *If N is normal in G, then HN/N is ss-quasinormal in G/N.*

(*c*) *If $H \leq F(G)$, then H is s-quasinormal in G.*

(*d*) *If H is a p-subgroup(p a prime), then H permutes with every Sylow q-subgroup of G with $q \neq p$.*

Proof. (a) and (b) are [3, Lemma 2.1], (c) is [3, Lemma 2.2], and (d) is [3, Lemma 2.5].

Lemma 2. ([6], Lemma 2.10) *Let H be a weakly s-permutable subgroup of a group G.*

(a) *If $H \leq K \leq G$, then H is weakly s-permutable in K.*

(b) *If N is normal in G and $N \leq H \leq G$, then H/N is weakly s-permutable in G/N.*

(c) *If H is a π-subgroup and N is a normal π'-subgroup of G, then HN/N is weakly s-permutable in G/N.*

(d) *Suppose H is a p-group for some prime p and H is not s-permutable in G. Then G has a normal subgroup M such that $|G : M| = p$ and $G = HM$.*

Lemma 3. ([6], Lemma 2.11) *Let N be an elementary abelian normal subgroup of a group G. Assume that N has a subgroup D such that $1 < |D| < |N|$ and every subgroup H of N satisfying $|H| = |D|$ is weakly s-permutable in G. Then some maximal subgroup of N is normal in G.*

Lemma 4. *Let N be an elementary abelian normal subgroup of a group G. Assume that N has a subgroup D such that $1 < |D| < |N|$ and every subgroup H of N satisfying $|H| = |D|$ is ss-quasinormal in G. Then some maximal subgroup of N is normal in G.*

Proof. By Lemma 3 and Lemma 1(c).

Lemma 5. ([1], III, 5.2 and IV, 5.4) *Suppose G is a group which is not p-nilpotent but whose proper subgroups are all p-nilpotent. Then*

(a) *G has a normal Sylow p-subgroup P for some prime p and $G = PQ$, where Q is a non-normal cyclic q-subgroup for some prime $q \neq p$.*

(b) *$P/\Phi(P)$ is a minimal normal subgroup of $G/\Phi(P)$.*

(c) *The exponent of P is p or 4.*

Lemma 6. *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If every subgroup of prime order or order 4(when P is a nonabelian 2-group) of P is either ss-quasinormal or weakly s-permutable in G, then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. By Lemmas 1(a) and 2(a), it is easy to see that G is a minimal non-p-nilpotent group. By Lemma 5, $G = [P]Q$. Let $x \in P$. Then the order of x is p or 4. By the hypothesis, $< x >$ is either ss-quasinormal or weakly s-permutable in G. If $< x >$

is ss-quasinormal in G, then $< x >$ is s-quasinormal in G by Lemma 1(c). If $< x >$ is weakly s-permutable in G, then there is a subnormal subgroup T of G such that $G =< x > T$ and

$$< x > \cap T \leq < x >_{sG} .$$

Hence

$$P = P \cap G = P \cap < x > T =< x > (P \cap T).$$

Since $P/\Phi(P)$ is abelian, we have $(P \cap T)\Phi(P)/\Phi(P)$ is normal in $G/\Phi(P)$. Since $P/\Phi(P)$ is the minimal normal subgroup of $G/\Phi(P)$, we have that $P \cap T \leq \Phi(P)$ or

$$P = (P \cap T)\Phi(P) = P \cap T.$$

If $P \cap T \leq \Phi(P)$, then $< x >= P$ is normal in G. It follows that G is p-nilpotent, a contraction. If $P = P \cap T$, then $T = G$ and so $< x >=< x >_{sG}$ is s-permutable in G. For any element x in P, now we have $< x > Q$ is a proper subgroup of G, then

$$< x > Q =< x > \times Q.$$

This implies that $G = P \times Q$, a contradiction.

Lemma 7. ([7], A, 1.2) *Let U, V, and W be subgroups of a group G. Then the following statements are equivalent:*

(a) $U \cap VW = (U \cap V)(U \cap W)$.

(b) $UV \cap UW = U(V \cap W)$.

Lemma 8. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P is ss-quasinormal in G, then G is p-nilpotent.*

Proof. This is a corollary of [3, Theorem 1.1].

Lemma 9. ([8], Lemma A.) *If P is an s-permutable p-subgroup of a group G for some prime p, then $N_G(P) \geq O^p(G)$.*

Lemma 10. ([9], X, 13) *Let G be a group and N is normal in G.*

(*a*) *If N is normal in G, then $F^*(N) \leq F^*(G)$.*

(*b*) *If $G \neq 1$, then $F^*(G) \neq 1$ and $F^*(G)/F(G) = Soc\ (F(G)C_G(F(G))/F(G))$.*

(*c*) *$F^*(F^*(G)) = F^*(G) \geq F(G)$. If $F^*(G)$ is Solvable, then $F^*(G) = F(G)$.*

3. Main results

Theorem 1. *Let P be a Sylow p-subgroup of a group G, where p is a prime divisor of $|G|$ with $(|G|, p-1) = 1$. If every maximal subgroup of P is either ss-quasinormal or weakly s-permutable in G, then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

(1) G has a unique minimal normal subgroup N. Moreover G/N is p-nilpotent, and $\Phi(G) = 1$.

Let N be a minimal normal subgroup of G. We shall prove that G/N satisfies the hypothesis of the theorem. Let M/N be a maximal subgroup of PN/N. Then $M = N(M \cap P)$. Let $P_1 = M \cap P$. It follows that $P_1 \cap N = M \cap P \cap N = P \cap N$ is a Sylow p-subgroup of N. Since

$$|P : P_1| = |P : M \cap P| = |PN : (M \cap P)N| = |PN/N : M/N| = p,$$

P_1 is a maximal subgroup of P. If P_1 is ss-quasinormal in G, then M/N is ss-quasinormal in G/N by Lemma 1(b). If P_1 is weakly s-permutable in G, then there is a subnormal subgroup T of G such that $G = P_1T$ and $P_1 \cap T \leq (P_1)_{sG}$. Thus

$$G/N = M/N \cdot TN/N = P_1N/N \cdot TN/N.$$

Since

$$(|N : P_1 \cap N|, |N : T \cap N|) = 1,$$

we have

$$(P_1 \cap N)(T \cap N) = N = N \cap G = N \cap P_1T.$$

By Lemma 7,

$$(P_1N) \cap (TN) = (P_1 \cap T)N.$$

It follows that

$$(P_1N/N) \cap (TN/N) = (P_1N \cap TN)/N = (P_1 \cap T)N/N \leq (P_1)_{sG}N/N \leq (P_1N/N)_{sG}.$$

Hence M/N is weakly s-permutable in G/N. Therefore, G/N satisfies the hypothesis of the theorem. The choice of G yields that G/N is p-nilpotent. Since the class of all p-nilpotent groups is a saturated formation, the uniqueness of N and the fact that $\Phi(G) = 1$ are obvious.

(2) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, then $N \leq O_{p'}(G)$ by Step (1). Since

$$G/O_{p'}(G) \cong (G/N)/(O_{p'}(G)/N)$$

is p-nilpotent, we have G is p-nilpotent, a contradiction.

(3) $O_p(G) = 1$.

If $O_p(G) \neq 1$, Step (1) yields $N \leq O_p(G)$ and

$$\Phi(O_p(G)) \leq \Phi(G) = 1.$$

Therefore, G has a maximal subgroup M such that $G = MN$ and $G/N \cong M$ is p-nilpotent. Since $O_p(G) \cap M$ is normalized by N and M, $O_p(G) \cap M$ is normal in G. The uniqueness of N yields $N = O_p(G)$. Clearly, $P = N(P \cap M)$. Furthermore $P \cap M < P$, thus there exists a maximal subgroup P_1 of P such that $P \cap M \leq P_1$. Hence $P = NP_1$. By the hypothesis, P_1 is either ss-quasinormal or weakly s-permutable in G. If we assume P_1 is ss-quasinormal in G, then P_1M_q is a group for $q \neq p$ by Lemma 1(d). Hence

$$P_1 < M_p, M_q | q \in \pi(M), q \neq p >= P_1M$$

is a group. Then $P_1M = M$ or G by maximality of M. If $P_1M = G$, then

$$P = P \cap P_1M = P_1(P \cap M) = P_1,$$

a contradiction. If $P_1M = M$, then $P_1 \leq M$. Therefore, $P_1 \cap N = 1$ and N is of prime order. Then the p-nilpotency of G/N implies the p-nilpotency of G, a contradiction. Therefore we may assume P_1 is weakly s-permutable in G. Then there is a subnormal subgroup T of G such that $G = P_1T$ and

$$P_1 \cap T \leq (P_1)_{sG} \leq O_p(G) = N \leq O^p(G)$$

because N is the unique minimal normal subgroup of G. Since $|G : T|$ is a power of p, $O^p(G) \leq T$. Hence,

$$P_1 \cap T \leq (P_1)_{sG} \leq O^p(G) \cap P_1 \leq T \cap P_1,$$

and so

$$P_1 \cap T = (P_1)_{sG} = O^p(G) \cap P_1.$$

Consequently, $G = PO^p(G)$ implies that $(P_1)_{sG}$ is normal in G by Lemma 9. By the minimality of N, we have $(P_1)_{sG} = N$ or $(P_1)_{sG} = 1$. If $(P_1)_{sG} = N$, then $N \leq P_1$ and $P = NP_1 = P_1$, a contradiction. Thus $P_1 \cap T = (P_1)_{sG} = 1$, and so $|T|_p = p$. Then T is p-nilpotent. Let $T_{p'}$ be the normal p-complement of T. Then $T_{p'}$ is subnormal in G and $T_{p'}$ is a p'-Hall subgroup of G. It follows that $T_{p'}$ is the normal p-complement of G, a contradiction.

(4) The final contradiction.

If P has a maximal subgroup P_1 which is weakly s-permutable in G, then there is a subnormal subgroup T of G such that $G = P_1T$ and

$$P_1 \cap T \leq (P_1)_{sG} \leq O_p(G) = 1.$$

Then $P_1 \cap T = 1$. Hence $|T|_p = p$. Therefore, T is p-nilpotent. Thus G is p-nilpotent, a contradiction. Now we may assume that all maximal subgroups of P

are ss-quasinormal in G. Then G is p-nilpotent by Lemma 8, a contradiction.

Theorem 2. *Let G be a group and P a Sylow p-subgroup of G, where p is the smallest prime dividing $|G|$. If P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is either ss-quasinormal or weakly s-permutable in G, then G is p-nilpotent.*

Proof. Suppose that the theorem is false and let G be a counterexample of minimal order. We will derive a contradiction in several steps.

(1) $O_{p'}(G) = 1$.

If $O_{p'}(G) \neq 1$, Lemmas 1(b) and 2(d) guarantee that $G/O_{p'}(G)$ satisfies the hypotheses of the theorem. Thus $G/O_{p'}(G)$ is p-nilpotent by the choice of G. Then G is p-nilpotent, a contradiction.

(2) $|D| > p$.

By Lemma 6.

(3) $|P : D| > p$.

By Theorem 1.

(4) P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is ss-quasinormal in G. Assume that $H \leq P$ such that $|H| = |D|$ and H is weakly s-permutable in G. Then there exists a subnormal subgroup T of G such that $G = HT$ and $H \cap K \leq H_{sG}$. By Lemma 2(d), we may assume G has a normal subgroup M such that $|G : M| = p$ and $G = HM$. Since $|P : D| > p$ by Step (3), M satisfies the hypotheses of the theorem. The choice of G yields that M is p-nilpotent. It is easy to see that G is p-nilpotent, contrary to the choice of G.

(5) If $N \leq P$ and N is minimal normal in G, then $|N| \leq |D|$.

Suppose that $|N| > |D|$. Since $N \leq O_p(G)$, N is elementary abelian. By Lemma 4, N has a maximal subgroup which is normal in G, contrary to the minimality of N.

(6) Suppose that $N \leq P$ and N is minimal normal in G. Then G/N is p-nilpotent.

If $|N| < |D|$, G/N satisfies the hypotheses of the theorem by Lemma 1(b). Thus G/N is p-nilpotent by the minimal choice of G. So we may suppose that $|N| = |D|$ by Step (5). We will show that every cyclic subgroup of P/N of order p or order 4(when P/N is a non-abelian 2-group) is ss-quasinormal in G/N. Let $K \leq P$ and $|K/N| = p$. By Step (2), N is non-cyclic, so are all subgroups containing N.

Hence there is a maximal subgroup $L \neq N$ of K such that $K = NL$. Of course, $|N| = |D| = |L|$. Since L is ss-quasinormal in G by the hypotheses, $K/N = LN/N$ is ss-quasinormal in G/N by Lemma 1(b). If $p = 2$ and P/N is non-abelian, take a cyclic subgroup X/N of P/N of order 4. Let K/N be maximal in X/N. Then K is maximal in X and $|K/N| = 2$. Since X is non-cyclic and X/N is cyclic, there is a maximal subgroup L of X such that N is not contained in L. Thus $X = LN$ and $|L| = |K| = 2|D|$. By the hypotheses, L is ss-quasinormal in G. By Lemma 1(b), $X/N = LN/N$ is ss-quasinormal in G/N. Hence G/N satisfies the hypotheses. By the minimal choice of G, G/N is p-nilpotent.

(7) $O_p(G) = 1$.
Suppose that $O_p(G) \neq 1$. Take a minimal normal subgroup N of G contained in $O_p(G)$. By Step (6), G/N is p-nilpotent. It is easy to see that N is the unique minimal normal subgroup of G contained in $O_p(G)$. Furthermore, $O_p(G) \cap \Phi(G) = 1$. Hence $O_p(G)$ is an elementary abelian p-group. On the other hand, G has a maximal subgroup M such that $G = MN$ and $M \cap N = 1$. It is easy to deduce that $O_p(G) \cap M = 1$, $N = O_p(G)$ and $M \cong G/N$ is p-nilpotent. Then G can be written as $G = N(M \cap P)M_{p'}$, where $M_{p'}$ is the normal p-complement of M. Pick a maximal subgroup S of $M_p = P \cap M$. Then $NSM_{p'}$ is a subgroup of G with index p. Since p is the minimal prime in $\pi(G)$, we know that $NSM_{p'}$ is normal in G. Now by Step (3) and the induction, we have $NSM_{p'}$ is p-nilpotent. Therefore, G is p-nilpotent, a contradiction.

(8) The minimal normal subgroup L of G is not p-nilpotent.
If L is p-nilpotent, then it follows that $L_{p'} \leq O_{p'}(G) = 1$ from the fact that $L_{p'}$ char L and L is normal in G. Thus L is a p-group. Then $L \leq O_p(G) = 1$ by Step (7), a contradiction.

(9) G is a non-abelian simple group.
Suppose that G is not a simple group. Take a minimal normal subgroup L of G. Then $L < G$. If $|L|_p > |D|$, then L is p-nilpotent by the minimal choice of G, contrary to Step (8). If $|L|_p \leq |D|$. Take $P_* \geq L \cap P$ such that $|P_*| = p|D|$. Hence P_* is a Sylow p-subgroup of P_*L. Since every maximal subgroup of P_* is of order $|D|$, every maximal subgroup of P_* is ss-quasinormal in G by hypotheses, thus in P_*L by Lemma 1(a). Now applying Theorem 1, we get P_*L is p-nilpotent. Therefore, L is p-nilpotent, contrary to Step (8).

(10) The final contradiction.
Suppose that H is a subgroup of P with $|H| = |D|$ and Q is a Sylow q-subgroup

with $q \neq p$. Then $HQ^g = Q^gH$ for any $g \in G$ by the hypotheses that H is ss-quasinormal in G and Lemma 1(d). Since G is simple by Step (9), $G = HQ$ from [1, VI, 4.10], the final contradiction.

Corollary 1. *Suppose that G is a group. If every non-cyclic Sylow subgroup P of G has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is either ss-quasinormal or weakly s-permutable in G, then G has a Sylow tower of supersolvable type.*

Theorem 3. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersoluble groups and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup P of E has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is either ss-quasinormal or weakly s-permutable in G. Then $G \in \mathcal{F}$.*

Proof. Since P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is either ss-quasinormal or weakly s-permutable in G by hypotheses, thus in E by Lemmas 1(a) and 2(a). Applying Corollary 1, we conclude that E has a Sylow tower of supersolvable type. Let q be the maximal prime divisor of $|E|$ and Q a Sylow q-subgroup of E. Then Q is normal in G. Since $(G/Q, E/Q)$ satisfies the hypotheses of the theorem, by induction, $G/Q \in \mathcal{F}$. For any subgroup H of Q with $|H| = |D|$, since $Q \leq O_q(G)$, H is either s-quasinormal or weakly s-permutable in G by Lemma 1(c). Since s-quasinormality implies weakly s-permutability and $F^*(Q) = Q$ by Lemma 10, we get $G \in \mathcal{F}$ by applying Theorem A.

Corollary 2. [3, Theorem 1.5) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersoluble groups. If there is a normal subgroup H of G such that $G/H \in \mathcal{F}$ and every maximal subgroup of any Sylow subgroup of H is ss-quasinormal in G, then $G \in \mathcal{F}$.*

Theorem 4. *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$, the class of all supersoluble groups and G a group with a normal subgroup E such that $G/E \in \mathcal{F}$. Suppose that every non-cyclic Sylow subgroup P of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is either ss-quasinormal or weakly s-permutable in G. Then $G \in \mathcal{F}$.*

Proof. We distinguish two cases:

Case 1. $\mathcal{F} = \mathcal{U}$.

Let G be a minimal counterexample.

(1) Every proper normal subgroup N of G containing $F^*(E)$ (if it exists) is

supersolvable.

If N is a proper normal subgroup of G containing $F^*(E)$, then $N/N \cap E \cong NE/E$ is supersolvable. By Lemma 10,

$$F^*(E) = F^*(F^*(E)) \leq F^*(E \cap N) \leq F^*(E),$$

so $F^*(E \cap N) = F^*(E)$. For any Sylow subgroup P of $F^*(E \cap N) = F^*(E)$, P has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is either ss-quasinormal or weakly s-permutable in G by hypotheses, thus in N by Lemmas 1(a) and 2(a). So N and $N \cap H$ satisfy the hypotheses of the theorem, the minimal choice of G implies that N is supersolvable.

(2) $E = G$.

If $E < G$, then $E \in \mathcal{U}$ by Step (1). Hence $F^*(E) = F(E)$ by Lemma 10. It follows that every Sylow subgroup of $F^*(E)$ is normal in G. By Lemma 1(c), every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is weakly s-permutable in G. Applying Theorem A for the special case $\mathcal{F} = \mathcal{U}$, $G \in \mathcal{U}$, a contradiction.

(3) $F^*(G) = F(G) < G$.

If $F^*(G) = G$, then $G \in \mathcal{U}$ by Theorem 3, contrary to the choice of G. So $F^*(G) < G$. By Step (1), $F^*(G) \in \mathcal{U}$ and $F^*(G) = F(G)$ by Lemma 10.

(4) The final contradiction.

Since $F^*(G) = F(G)$, each non-cyclic Sylow subgroup of $F^*(G)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is weakly s-permutable in G by Lemma 1(c). Applying Theorem A, $G \in \mathcal{U}$, a contradiction.

Case 2. $\mathcal{F} \neq \mathcal{U}$.

By hypotheses, every non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order $|H| = |D|$ or with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is either ss-quasinormal or weakly s-permutable in G, thus in E by Lemmas 1(a) and 2(a). Applying Case 1, $E \in \mathcal{U}$. Then $F^*(E) = F(E)$ by Lemma 10. It follows that each Sylow subgroup of $F^*(E)$ is normal in G. By Lemma 1(c), each non-cyclic Sylow subgroup of $F^*(E)$ has a subgroup D such that $1 < |D| < |P|$ and every subgroup H of P with order

$|H| = |D|$ or with order $2|D|$ (if P is a non-abelian 2-group and $|P : D| > 2$) is weakly s-permutable in G. Applying Theorem A, $G \in \mathcal{F}$.

Corollary 3.[10, Theorem 3.3] *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. Then $G \in \mathcal{F}$ if and only if every maximal subgroup of any Sylow subgroup of $F^*(H)$ is ss-quasinormal in G.*

Corollary 4.[10, Theorem 3.7] *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. Then $G \in \mathcal{F}$ if and only if every cyclic subgroup of any Sylow subgroup of $F^*(H)$ of prime order or order 4 is ss-quasinormal in G.*

Corollary 5.[11, Theorem 3.1) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of any Sylow subgroup of $F^*(H)$ are c-normal in G, then $G \in \mathcal{F}$.*

Corollary 6.[11, Theorem 3.2) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If every cyclic subgroup of any Sylow subgroup of $F^*(H)$ of prime order or order 4 is c-normal in G, then $G \in \mathcal{F}$.*

Corollary 7.[12, Theorem 3.4) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If all maximal subgroups of any Sylow subgroup of $F^*(H)$ are s-quasinormal in G, then $G \in \mathcal{F}$.*

Corollary 8.[13, Theorem 3.3) *Let $\mathcal{F}$ be a saturated formation containing $\mathcal{U}$. Suppose that G is a group with a normal subgroup H such that $G/H \in \mathcal{F}$. If every cyclic subgroup of any Sylow subgroup of $F^*(H)$ of prime order or order 4 is s-quasinormal in G, then $G \in \mathcal{F}$.*

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Changwen Li
School of Mathematical Science, Xuzhou Normal University
Xuzhou 221116, Jiangsu, People's Republic of China
email:*lcwxz@xznu.edu.cn*