Acta Universitatis Apulensis No. 25/2011
ISSN: 1582-5329 pp. 105-111

ON THE METRIZATION

Mohammad Azram

Abstract. In this article, we have studied the concept, property and interrelationship of being T_1-space, discrete space, metrizable space, separable space and topologically totally bounded space with finite, infinite and countable underlying set. With some additional conditions we have established equivalence between T_1-space, discrete space, metrizable space, separable space and topologically totally bounded space. At the end we have extended the same to a disjoint (free) union.

2000 *Mathematics Subject Classification*: 54A05.

1. Introductions

It is a problem of considerable interest to determine what types of topological spaces are metrizable [1, 2, 4-7]. An important class of topological spaces that is metrizable spaces, which play a fundamental role are usually introduced at a later stage of a topological course to enjoy the interplay of the concept of countability, separability and Lindelof. Most of the initial courses on general/point-set topology do not include especially the proof of well-celebrated metrization theorem of Urysohn. It will be quite useful to have the knowledge of metrizability at an early stages on a finite, countable, infinite, countable union and free union of topological spaces. Focus should be on basic concepts with examples of topological spaces, as an expeditious means of involving mathematics loving students in creative thinking and research. Although the underlying/ground set always plays a pivotal role in building a topological structure but most important is the different organizations of members of ground/underlying set. In this article, we will study the concept and property of being T_1-space, discrete space, metrizable space, separable space and topologically totally bounded space with finite, infinite and countable underlying set. With some additional conditions we will establish equivalence between T_1-space, discrete space, metrizable space, separable space and topologically totally bounded space. At the end we will extend the same to disjoint (free) union.

2.Material and Methods

Terminology, definition and concept about discrete/indiscrete space, compact space, 2^{nd} countable space, dense set, Euclidean metric and Minkowskis inequality will be the usual one [2, 4, 5]. A topological space X will be called T_1-space if each singleton subset of X is closed in X. Similar to that of 0-dimensional space and/or almost discrete space [3], we will consider a topological space, a closed topological space if each closed set is also open. A metric space $(X,\ d)$ is bounded if and only if $\sup\{d(x,y)|x,y \in X\}$ is finite. A topological space $(X,\ T)$ is metrizable if and only if there exist a metric d on X such that the topology induced by d coincides with T. A metric d for a metrizable space X is totally bounded if for each $\epsilon > 0$, the open covering $\left\{S_e^d(x)|x \in X\right\}$ of X has a finite sub-covering. A topological space will be called separable if it contains a countable dense subset. A topological space $(X,\ T)$ will be called topologically totally bounded if and only if there exist a metric d on X such that the topology induced by d coincides with T and $(X,\ d)$ is totally bounded. A topological space X will be considered as disjoint (free) union of metrizable spaces if $X = \bigcup\limits_{\alpha\in\Lambda} X_\alpha \ni$ for $\alpha \neq \beta$, $X_\alpha \bigcap X_\beta = \phi$ and (X_α,T_α) is metric topology induced by the metric $d_\alpha\,\forall\alpha \in \Lambda$.

3.Results and discussion

Consider $X = \{1, 2, 3\}$ and $\tau = \{\phi, X, \{1\}, \{2, 3\}\}$. It can be observed that a closed topological space may not necessarily be a discrete space, indiscrete space, or a T_1-space. A finite T_1-space is always a discrete space but $X = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, ...\right\}$ with usual topology indicates that the case is not true if X is countable or infinite but;

Lemma 1. *A closed topological space is a T_1-space if and only if it is discrete space*

Proof. Trivial.

Every discrete space is surely metrizable (discrete metric) but only finite metrizable spaces will certainly be discrete. Metrizable spaces satisfy the separation axioms. An infinite or even a countable metrizable space may not be discrete. As an example, one can consider $X = \left\{0, 1, \frac{1}{2}, \frac{1}{3}, ...\right\}$ with usual topology. A discrete space is separable if and only if it is countable. Real line with its usual topology gives us an example to say a discrete space may not be separable but;

Lemma 2. *A countable discrete space is always metrizable and separable.*

Proof. Trivial.

Every metric is equivalent to a bounded metric but a bounded metric may not be totally bounded. $d(x,y)=\min\left\{\hat{d}(x,y),1\right\}$ where $\hat{d}$ is the Euclidean metric defined on $\mathbb{R}^1$ is bounded but not totally bounded. Mostly the totally bounded metric is linked with 2^{nd} countable spaces and compact spaces. Here we will try to link it with separable spaces. In the next few lines, we will establish a pivotal result, which will enable us to link T_1-spaces, discrete spaces, metrizable spaces and separable spaces.

Theorem 1. *A topological space is metrizable, separable if and only if it is topologically totally bounded.*

Proof.

$\Rightarrow$Assume without loss of generality that the topological space (X,τ) is equivalent to (X,d) where d is bounded metric on X such that $d(x,y)\leq 1\ \forall x,y\in X$. Define $\hat{d}$ on X as $\hat{d}(x,y)=\sqrt{\sum\limits_{i=1}^{\infty}\left(\frac{d(x,x_i)-d(y,x_i)}{i}\right)^2}$ where $x_i\in D$ and $D=\{x_1,x_2,x_3,...\}$ is a countable dense subset of X.
Firstly, we claim that $\hat{d}$ is a metric on X. Clearly $\hat{d}(x,y)>0$ & $\hat{d}(x,y)=\hat{d}(y,x)$ $\forall x,y\in X$.
Now;
If $x=y$ than $\hat{d}(x,y)=0$ (obvious).
On the other hand let $\hat{d}(x,y)=0$ but $x\neq y$ then $d(x,x_i)=d(y,x_i)\,\forall i\Rightarrow\exists x_{i_0}\in D\ni d(x,x_{i_0})<\frac{d(x,y)}{3}\Rightarrow d(y,x_{i_0})\geq d(x,y)-d(x,x_{i_0})>d(x,y)-\frac{d(x,y)}{3}=\frac{2}{3}d(x,y)\Rightarrow d(y,x_{i_0})\neq d(x,x_{i_0})\Rightarrow$ Contradiction. Hence, $\hat{d}(x,y)=0\Leftrightarrow x=y$.
Triangle inequality is direct consequence of Minkowskis inequality

$$\begin{aligned}\hat{d}(x,z)+\hat{d}(z,y) &= \sqrt{\sum_{i=1}^{\infty}\left(\frac{d(x,x_{i_0})-d(z,x_{i_0})}{i}\right)^2}+\sqrt{\sum_{i=1}^{\infty}\left(\frac{d(z,x_{i_0})-d(y,x_{i_0})}{i}\right)^2}\\ &\geq \sqrt{\sum_{i=1}^{\infty}\left(\frac{d(x,x_{i_0})-d(y,x_{i_0})}{i}\right)^2}=\hat{d}(x,y)\end{aligned}$$

Now we claim that d and $\hat{d}$ are equivalent metric. Since $d(x,y)\leq 1\ \forall x,y,\in X$ $\therefore$ For each $x,y\in X$ & $\forall\epsilon>0\ \exists N\ni\sum\limits_{i=N+1}^{\infty}\left(\frac{d(x,x_i)-d(y,x_i)}{i}\right)^2<\frac{\epsilon^2}{4}$.

Since

$$|d(x,x_i)-d(y,x_i)| \le d(x,y) \therefore d(x,y) < \frac{\epsilon}{\sqrt{4N}}$$
$$\Rightarrow \hat{d}(x,y) = \sqrt{\sum_{i=1}^{N}\left(\frac{d(x,x_i)-d(y,x_i)}{i}\right)^2} + \sqrt{\sum_{i=N+1}^{\infty}\left(\frac{d(x,x_i)-d(y,x_i)}{i}\right)^2}$$
$$< \frac{\epsilon}{\sqrt{2}} < \epsilon$$

Similarly; if $\forall x \in X \exists y_n \in X \ni \hat{d}(x,y_n) \to 0$ as $n \to \infty$ then $|d(x,x_i)-d(y_n,x_i)| \to 0$ as $n \to \infty$ $\forall i \Rightarrow d(x,y_n) \to 0$ as $n \to \infty$.
Because otherwise $\exists \epsilon_0 > 0 \& \{y_{n_k}\} \in \{y_n\} \ni d(x,y_{n_k}) > \epsilon_o \, \forall n_k$. Choosing $x_{m_i} \Longrightarrow x$ w.r.t. d as $m_i \to \infty \Rightarrow$ for each $n_k, |d(x,x_{m_i})-d(y_{n_k},x_{m_i})| \to d(y_{n_k},x) > \epsilon_0$ as $m_i \to \infty \Rightarrow$ contradiction to $|d(x,x_i)-d(y_n,x_i)| \to 0$ as $n \to \infty \, \forall i$.
Hence d is equivalent to $\hat{d}$.
To show that X is topologically totally bounded, it is sufficient to show that $(X,\hat{d})$ is totally bounded.
Define

$$\Phi : X \to \Phi(X) \subset \prod_{i=1}^{\infty} I_i \, (I_i \approx I) \text{ as } x \to \{d(x,x_i)\}.$$

Define $\hat{d}$ on $\Phi(X) \ni \hat{d}(\Phi(x),\Phi(y)) = \hat{d}(x,y)$.
Now

$$\Phi(x) = \Phi(y) \Leftrightarrow \hat{d}(\Phi(x),\Phi(y)) = \hat{d}(x,y) = 0 \Leftrightarrow x = y \Rightarrow \Phi \text{ is } 1-1.$$

Since;

$$\hat{d}(x,y_n) \to 0 \text{ as } n \to \infty \Leftrightarrow d(x,y_n) \to 0 \text{ as } n \to \infty \Leftrightarrow$$
$$\hat{d}(\Phi(x),\Phi(y)) \to 0 \text{ as } n \to \infty \Rightarrow \Phi \,\&\, \Phi^{-1} \text{are continuous.}$$

Hence, $\Phi(X)$ is embedding of X into $\prod_{i=1}^{\infty} I_i$ via Φ.

Now;

$$n \in Z^+, \, X \subseteq \bigcup_{i=1}^{\infty} S_{1/n}^{\hat{d}}(x_i) \Rightarrow \Phi(X) \subseteq \bigcup_{i=1}^{\infty} S_{1/n}^{\hat{d}} \Phi(x_i)$$

which implies the existence of the finite sub-cover for $\Phi(X)$ that is

$$\Phi(X) \subseteq \left[\bigcup_{m_i=1}^{N} S^{\hat{d}}_{1/n}\Phi(x_{m_i})\right] \bigcap \Phi(X) = \bigcup_{m_i=1}^{N} \Phi\left[S^{\hat{d}}_{1/n}(x_{m_i})\right]$$

$$\Rightarrow X \subseteq \bigcup_{m_i=1}^{N}\left[S^{\hat{d}}_{1/n}(x_{m_i})\right]$$

which implies $(X,\,\hat{d})$ is totally bounded.
$\Leftarrow$ (Conversely)
Let X be topologically bounded

$$\Rightarrow \quad \exists \text{ metric } d \ni \ \forall n \in Z^+ \, \exists\, x_{1_k}, x_{2_k}, ..., x_{n_k} \in X \ \ni X = \bigcup_{i=1}^{k_n} S^{d}_{1/n}(x_{i_n}) \forall x \in X$$

$$\& \forall n \in Z^+, x \in S^{d}_{1/n}(x_{i_n}) \text{ for some } i_n$$

$$\Rightarrow x_{i_n} \to x \text{ as } n \to \infty \Rightarrow x \in \bar{D} = \overline{\{x_{i_n} | n = 1, 2, ... \& i = 1, 2, .., k_n\}} \Rightarrow X \subseteq \bar{D}$$

which implies that D is dense in X and consequently X is separable. This completes the proof. $\diamondsuit$

Theorem 2. *If X is a countable space with closed topology then the following are equivalent.*

a) *X is T_1-space.*

b) *X is discrete space.*

c) *X is metrizable, separable space.*

d) *X is topologically totally bounded.*

Now, we will investigate whether or not theorem 2 *is valid if X is disjoint (free) union of countable spaces with closed topologies. I suspect, metrizability is a crucial property, needs to be investigated. In the next few lines we will establish a more generalized result about metrizable spaces.*

Theorem 3. *Disjoint (free) union of metrizable spaces is metrizable.*

Proof. Let X be disjoint (free) union of metrizable spaces that is $X = \bigcup_{\alpha\in\Lambda} X_\alpha \ni$ for $\alpha \neq \beta$, $X_\alpha \cap X_\beta = \phi$ and $\forall \alpha$, (X_α, T_α) is metric topology induced by the metric $d_\alpha \, \forall \alpha \in \Lambda$. Now, define the metric $\hat{d}_\alpha(x, y) = \{d_\alpha(x, y), 1\}$. It is straightforward to observe that $\hat{d}_\alpha$ and d_α are equivalent.

Define;

$$d(x,y)=\begin{cases}\hat{d}_\alpha(x,y) & \text{if} \quad x,y\in X_\alpha \\ 1 & \text{if} \quad x\in X_\alpha, y\in X_\beta \ \& \ \alpha\neq\beta\end{cases}$$

Clearly $d(x,y)=0\Leftrightarrow x=y$ and $d(x,y)\geq 0$.
Now;
If $x,y\in X_\alpha$, $z\in X_\beta\,(\alpha\neq\beta)$ then $d(x,y)\leq 1<1+1=d(x,z)+d(z,y)$.
If $x\in X_\alpha$, $y\in X_\beta$ then;

$$d(x,y)=1\leq\begin{cases}1\leq d(x,z)+d(z,y) & z\in X_\alpha \text{ or } X_\beta \\ 2=d(x,z)+d(z,y) & z\in X_\gamma\,,\gamma\neq\alpha,\beta\end{cases}$$

If $x,y,z\in X_\alpha$ then $d|_{X_\alpha}=\hat{d}_\alpha\Rightarrow d(x,y)\leq d(x,z)+d(z,y)$. Hence d is a metric on $X=\bigcup\limits_{\alpha\in\Lambda}X_\alpha$.
Now, we need to show that $(X,\,d)$ is the same as the disjoint union topology (X,T).
It is sufficient to show that $B=\left\{S_\epsilon^d(x)|\epsilon<1,\,x\in X\right\}$ is a basis of (X,T).

$$\forall S_\epsilon^d\in B,\ \text{since } \epsilon<1 \text{ so } N(x,\epsilon)\subset X_\alpha,$$
$$\text{if } x\in X_\alpha \,\&\, S_\epsilon^d(x)\in T_\alpha\subset T\Rightarrow B\subset T.$$

Now $\forall A\subset T\,\&\,\forall x\in A\subset\bigcup\limits_\alpha X_\alpha$.
We need to find a $S_\alpha^d(x)\in B\ni x\in S_\alpha^d(x)\subset A$. If $x\in X_\alpha$ then

$$A\cap X_\alpha\in T_\alpha \text{ so } \exists\, S_\epsilon^{d_\alpha}(x) \text{ (here } \epsilon<1 \text{ in } T_\alpha)\ \ni x\in\ S_\epsilon^{d_\alpha}(x)\subset A\cap X_\alpha\subset A.$$

It is straightforward to see that $S_\epsilon^{d_\alpha}(x)=S_\epsilon^{\hat{d}_\alpha}(x)=S_\epsilon^d(x)$ $(\because\ \epsilon<1)$. Hence, disjoint (free) union of metrizable spaces is metrizable. This completes the proof. $\diamondsuit$

References

[1] Bing, R.H., *Metrization of Topological Spaces*, Canadian Journal of Math., Vol.3 (1951), PP: 175-186.

[2] Dugundji, J., Topology, Boston. Allyn and Bacon (1966).

[3] Karno, Z., *On Discrete and Almost DiscreteTopological Spaces*, Formalized Mathematics, Vol. 3 (2) (1992), PP: 305-310.

[4] Kelley, J.L., General Topology, New York, Springer-Verlag (1975).

[5] Munkres, J.R., Topology, Prentice Hall (1999)

[6] Nagata, J., *On a necessary and Sufficient condition of Metrizability*, J. Inst. Poly. Osaka City Uni., Vol.1 (1950), PP: 93-100

[7] Smirnov, Yu. M., *On Metrization of Topological Spaces*, Amer. Math. Soc. Transl Ser., Vol.1 (1953), PP: 91

Mohammad Azram
Department of Science in Engineering (SIE)
University of IIUM
Address: Jalan Gombak, 50728, KL, Malaysia
email:*azram50@hotmail.com*