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# A NEW CLASS OF MEROMORPHIC FUNCTIONS

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Abstract. In this paper, making use of a linear operator we introduce and study a new class of meromorphic functions. We derive some inclusion relations and a radius problem. This class contain many known classes of meromorphic functions as special cases.

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## 1. Introduction

Let $M$ denotes the class of functions of the form

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n\, z^n, \tag{1.1}$$

which are analytic in the punctured open unit disc $D = \{z : 0 < |z| < 1\}$. Further let $P_k(\alpha)$ be the class of functions $p(z), z \in E$, analytic in $E = D \cup \{0\}$ satisfying $p(0) = 1$ and

$$\int_0^{2\pi} \left| \frac{\operatorname{Re} p(z) - \alpha}{1 - \alpha} \right| d\theta \leq k\pi, \tag{1.2}$$

where $z = re^{i\theta}, k \geq 2,\ 0 \leq \alpha < 1$. This class was introduced by Padmanbhan and Paravatham [5]. For $\alpha = 0$ we obtain the class $P_k$ defined by Pinchuk [6] and $P_2(\alpha) = P(\alpha)$ is the class with positive real part greater than $\alpha$. Also $p \in P_k(\alpha)$, if and only if

$$p(z) = \left(\frac{k}{4} + \frac{1}{2}\right) p_1(z) - \left(\frac{k}{4} - \frac{1}{2}\right) p_2(z), \tag{1.3}$$

where $p_1,\ p_2 \in P(\alpha)$, $z \in E$. The class $M$ is closed under then the convolution or Hadamard product denoted and defined by

$$(f * g)(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n\, b_n\, z^n, \tag{1.4}$$

where

$$f(z) = \frac{1}{z} + \sum_{n=0}^{\infty} a_n\ z^n,\ \ g(z) = \frac{1}{z} + \sum_{n=0}^{\infty} b_n\ z^n.$$

The incomplete Beta function is defined by

$$\phi(a,c;z) = \frac{1}{z} + \sum_{n=0}^{\infty} \frac{(a)_{n+1}}{(c)_{n+1}}\ z^n, \qquad a,c \in R, c \neq 0,-1,-2,...,z \in D \tag{1.5}$$

where $(a)_n$ is the Pochhammer symbol. Using $\phi(a,c;z)$ Liu and Srivastava [3] defined an operator $\pounds(a,c) : M \to M$, as

$$\pounds(a,c)f(z) = \phi(a,c;z) * f(z). \tag{1.6}$$

This operator is closely related to the Carlson-Shaffer operator studied in [1]. Analogous to $\pounds(a,c)$, in [2] Cho and Noor defined $I_\mu(a,c) : M \to M$ as

$$I_\mu(a,c)f(z) = (\phi(a,c;z))^{-1} * f(z), \ \ (\mu > 0, a > 0,\ c \neq -1,-2,-3,...,\ \ z \in D). \tag{1.7}$$

We note that

$$I_2(2,1)f(z) = f(z), \quad \text{and} \quad I_2(1,1)f(z) = zf'(z) + 2f(z)$$

Using (1.7), it can be easily verified that

$$z(I_\mu(a+1,c)f(z))' = aI_\mu(a,c)f(z) - (a+1)I_\mu(a+1,c)f(z) \tag{1.8}$$

and

$$z(I_\mu(a,c)f(z)' = \mu I_{\mu+1}(a,c)f(z) - (\mu+1)I_\mu(a,c)f(z). \tag{1.9}$$

Furthermore for $f \in M$, $\operatorname{Re} b > 0$ the Generalized Bernadi Operator is defined as

$$J_b f(z) = \frac{b}{z^{b+1}} \int_0^z t^b f(t)\ dt. \tag{1.10}$$

Using (1.10) it can easily be verified that

$$z(I_\mu(a,c)J_b f(z))' = bI_\mu(a,c)f(z) - (b+1)I_\mu(a,c)J_b f(z). \tag{1.11}$$

Now using the operator $I_\mu(a,c)$, we define the following class of meromorphic functions.

**Definition 1.1** *Let $f \in M$, then $f(z) \in Q_k^{\mu}(a,c,\lambda,\alpha)$, if and only if*

$$-\frac{z\left(I_\mu(a,c)f(z)\right)' + \lambda z^2\left(I_\mu(a,c)f(z)\right)''}{(1-\lambda)I_\mu(a,c)f(z) + \lambda z\left(I_\mu(a,c)f(z)\right)'} \in P_k(\alpha),$$

*where* $k \geq 2,\ 0 \leq \lambda \leq 1,\ 0 \leq \alpha < 1,\ \mu > 0, a > 0,\ c \neq -1,-2,-3,...,\ \ z \in D.$

**Special Cases:**

(i) For $\lambda = 0$ and $\lambda = 1$ this class was already discussed by Noor in [2].

(ii) For $\lambda = 0,\ \mu = 2, a = 2, c = 1, k = 2 - \frac{zf'(z)}{f(z)} \in P(\alpha)$.

(iii) For $\lambda = 1,\ \mu = 2, a = 2, c = 1, k = 2$

$$-\frac{[zf'(z)]'}{f'(z)} \in P(\alpha).$$

## 2. Preliminary Results

**Lemma 2.1 [4].** *Let $u = u_1 + iu_2$ and $v = v_1 + iv_2$ and let $\varphi(u,v)$ be a complex valued function satisfying the conditions:*

*i)* $\varphi(u,v)$ *is continuous in* $D \subset C^2$,

*ii)* $(1,0) \in D$ *and* $\operatorname{Re}\varphi(1,0) > 0$,

*iii)* $\operatorname{Re}\varphi(iu_2, v_1) \leqslant 0$ *whenever* $(iu_2, v_1) \in D$ *and* $v_1 \leqslant -\frac{1}{2}(1+u_2^2)$.

If $h(z)$ is a function analytic in $D \cup \{0\}$ such that $(h(z), zh'(z)) \in D$ and $\operatorname{Re}\varphi(h(z), zh'(z)) > 0$ for $z \in D \cup \{0\}$, then $\operatorname{Re} h(z) > 0$ in $D \cup \{0\}$.

## 3. Main Results

**Theorem 3.1.** *For $k \geq 2,\ 0 \leq \lambda \leq 1,\ 0 \leq \alpha < 1,\ \mu > 0, a > 0,\ c \neq -1,-2,-3,...,\ \ z \in D$.*

$$Q_k^{\mu+1}(a,c,\lambda,\alpha) \subset Q_k^{\mu}(a,c,\lambda,\beta) \subset Q_k^{\mu}(a+1,c,\lambda,\gamma).$$

*Proof.* First we prove that

$$Q_k^{\mu+1}(a,c,\lambda,\alpha) \subset Q_k^{\mu}(a,c,\lambda,\beta).$$

Let $f(z) \in Q_k^{\mu+1}(a,c,\lambda,\alpha)$ and set

$$-\frac{z\left(I_\mu(a,c)f(z)\right)' + \lambda z^2(I_\mu(a,c)f(z))''}{(1-\lambda)I_\mu(a,c)f(z) + \lambda z\left(I_\mu(a,c)f(z)\right)'} = H(z). \tag{3.1}$$

From (1.9) and (3.1), we have

$$\frac{\mu[\lambda z\,(I_{\mu+1}(a,c)f(z))' + (1-\lambda)I_{\mu+1}(a,c)f(z)]}{(1-\lambda)I_{\mu}(a,c)f(z)+\lambda z\,(I_{\mu}(a,c)f(z))'} = -H(z)+(\mu+1). \tag{3.2}$$

After multiplying (3.2) by $z$ and then by logarithmic differentiation , we obtain

$$-\frac{z\,(I_{\mu+1}(a,c)f(z))' + \lambda z^2(I_{\mu+1}(a,c)f(z))''}{(1-\lambda)I_{\mu+1}(a,c)f(z)+\lambda z\,(I_{\mu+1}(a,c)f(z))'} = H(z)+\frac{zH'(z)}{-H(z)+(\mu+1)} \in P_k(\alpha).$$

Let

$$\varphi_\mu(z) = \frac{1}{\mu+1}\left[\frac{1}{z}+\sum_{k=0}^{\infty} z^k\right] + \frac{\mu}{\mu+1}\left[\frac{1}{z}+\sum_{k=0}^{\infty} k\ z^k\right],$$

then

$$\begin{aligned}
H(z)*z\varphi_\mu(z) &= H(z)+\frac{zH'(z)}{-H(z)+(\mu+1)} \\
&= \left(\frac{k}{4}+\frac{1}{2}\right)\big(h_1(z)+z\varphi_\mu(z)\big) - \left(\frac{k}{4}-\frac{1}{2}\right)\big(h_2(z)+z\varphi_\mu(z)\big) \\
&= \left(\frac{k}{4}+\frac{1}{2}\right)\left\{h_1(z)+\frac{zh_1'(z)}{-h_1(z)+(\mu+1)}\right\} - \\
&- \left(\frac{k}{4}-\frac{1}{2}\right)\left\{h_1(z)+\frac{zh_2'(z)}{-h_2(z)+(\mu+1)}\right\}.
\end{aligned}$$

As $f(z)\in Q_k^{\mu+1}(a,c,\lambda,\alpha)$, so

$$h_i(z)+\frac{zh_i'(z)}{-h_i(z)+(\mu+1)} \in P(\alpha) \qquad i=1,2.$$

Let $h_i(z)=(1-\beta)p_i(z)+\beta$, then

$$\left[(1-\beta)p_i(z)+\frac{(1-\beta)zp_i'(z)}{-(1-\beta)p_i(z)+(\mu-\beta+1)}+(\beta-\alpha)\right]\in P.$$

We want to show that $p_i\in P$, for $i=1,2$. For this we formulate a functional $\varphi(u,v)$ by taking $u=p_i(z)$ and $v=zp_i'(z)$ as follows:

$$\varphi(u,v)=(1-\beta)u+\frac{(1-\beta)v}{-(1-\beta)u+(\mu-\beta+1)}+(\beta-\alpha).$$

The first two conditions of Lemma 2.1 are clearly satisfied. For the third condition we proceed as follows:

$$\operatorname{Re}\varphi(iu_2,v_1)=(\beta-\alpha)+\frac{(1-\beta)v_1}{(1-\beta)^2u_2^2+(\mu-\beta+1)^2}.$$

When $v_1 \leq -\frac{1}{2}(1+u_2^2)$, then

$$\begin{aligned} \operatorname{Re}\varphi(iu_2, v_1) &\leq (\beta-\alpha) - \frac{(1-\beta)(\mu-\beta+1)(1+u_2^2)}{2[(1-\beta)^2 u_2^2 + (\mu-\beta+1)^2]} \\ &= \frac{A+Bu_2^2}{2C}, \end{aligned}$$

where

$$\begin{aligned} A &= (\mu-\beta+1)\left\{2(\beta-\alpha)(\mu-\beta+1)-(1-\beta)\right\}, \\ B &= (1-\beta)\left\{2(\beta-\alpha)(1-\beta)-(\mu-\beta+1)\right\}, \\ C &= (1-\beta)^2 u_2^2 + (\mu-\beta+1)^2. \end{aligned}$$

We note that $\operatorname{Re}\varphi(iu_2, v_1) \leq 0$ if and only if $A \leq 0,$ and $B \leq 0$. From $A \leq 0$, we have

$$\beta = \frac{1}{4}\left[(3+2\mu+2\alpha) - \sqrt{(3+2\mu+2\alpha)^2 - 8(2\alpha+2\alpha\mu+1)}\right],$$

and $B \leq 0$ gives us $0 \leq \beta < 1$. Hence by Lemma 2.1, $p_i \in P$, for $i = 1,2$ and consequently, $f(z) \in Q_k^{\mu}(a,c,\lambda,\beta)$. Similarly, we can prove the other inclusion.

**Theorem 3.2.** *If* $f(z) \in Q_k^{\mu+1}(a,c,\lambda,\alpha)$ *and* $J_b$ *is given by* (1.11) *then* $J_b f(z) \in Q_k^{\mu+1}(a,c,\lambda,\beta)$.

*Proof.* Let $f(z) \in Q_k^{\mu+1}(a,c,\lambda,\alpha)$ and set

$$-\frac{z\left(I_\mu(a,c)J_b f(z)\right)' + \lambda z^2 \left(I_\mu(a,c)J_b f(z)\right)''}{(1-\lambda)I_\mu(a,c)J_b f(z) + \lambda z\left(I_\mu(a,c)J_b f(z)\right)'} = H(z). \tag{3.3}$$

Using (1.11), (3.3) and after some simplifications, we obtain

$$-\frac{z\left(I_\mu(a,c)f(z)\right)' + \lambda z^2 \left(I_\mu(a,c)f(z)\right)''}{(1-\lambda)I_\mu(a,c)f(z) + \lambda z\left(I_\mu(a,c)f(z)\right)'} = H(z) + \frac{zH'(z)}{-H(z)+(b+1)}. \tag{3.4}$$

Now working as in Theorem 3.1 we obtain the desired result.

**Theorem 3.3.** *If* $f(z) \in Q_k^{\mu}(a,c,\lambda,\alpha)$ *then* $f(z) \in Q_k^{\mu+1}(a,c,\lambda,\alpha)$, *for* $|z| < r_0$, *where*

$$r_0 = \frac{1}{4}\{\sqrt{4+\mu(\mu+2)} - 2\}. \tag{3.5}$$

*Proof.* Since $f(z) \in Q_k^{\mu}(a,c,\lambda,\alpha)$, so working in the same way as in Theorem 3.1, we have

$$-\frac{z\left(I_\mu(a,c)f(z)\right)' + \lambda z^2 \left(I_\mu(a,c)f(z)\right)''}{(1-\lambda)I_\mu(a,c)f(z) + \lambda z\left(I_\mu(a,c)f(z)\right)'} = H(z) \tag{3.6}$$

$$= \left(\frac{k}{4}+\frac{1}{2}\right)h_1(z) - \left(\frac{k}{4}-\frac{1}{2}\right)h_2(z),$$

where $h_i \in P(\alpha)$ for $i = 1, 2$.
From (1.9) , (3.6) and after some simplification, we have

$$-\frac{z\left(I_{\mu+1}(a,c)f(z)\right)' + \lambda z^2\left(I_{\mu+1}(a,c)f(z)\right)''}{(1-\lambda)I_{\mu+1}(a,c)f(z) + \lambda z\left(I_{\mu+1}(a,c)f(z)\right)'}$$

$$\begin{aligned} = \quad & \left(\frac{k}{4}+\frac{1}{2}\right)\left\{h_1(z) + \frac{zh_1'(z)}{-h_1(z)+(\mu+1)}\right\} \\ & -\left(\frac{k}{4}-\frac{1}{2}\right)\left\{h_2(z) + \frac{zh_2'(z)}{-h_2(z)+(\mu+1)}\right\}. \end{aligned} \tag{3.7}$$

Let $h_i(z) = (1-\alpha)p_i(z) + \alpha$. Then (3.7) becomes

$$\frac{1}{1-\alpha}\left[-\frac{z\left(I_{\mu+1}(a,c)f(z)\right)' + \lambda z^2\left(I_{\mu+1}(a,c)f(z)\right)''}{(1-\lambda)I_{\mu+1}(a,c)f(z) + \lambda z\left(I_{\mu+1}(a,c)f(z)\right)'} - \alpha\right]$$

$$= \left(\frac{k}{4}+\frac{1}{2}\right)\left\{p_1(z) + \frac{zp_1'(z)}{-h_1(z)+(\mu+1)}\right\} - \left(\frac{k}{4}-\frac{1}{2}\right)\left\{p_2(z) + \frac{zp_2'(z)}{-p_2(z)+(\mu+1)}\right\}. \tag{3.8}$$

Now consider

$$\operatorname{Re}\left[h_i(z) + \frac{zh_i'(z)}{-h_i(z)+(\mu+1)}\right] \geq \operatorname{Re} h_i(z) - \left|\frac{zh_i'(z)}{-h_i(z)+(\mu+1)}\right|.$$

Using the well known distorsion bounds for the class $P$, we have

$$\operatorname{Re}\left[h_i(z) + \frac{zh_i'(z)}{-h_i(z)+(\mu+1)}\right] \geq \operatorname{Re} h_i(z)\left[1 + \frac{2r}{1-r^2}\frac{1}{\frac{1+r}{1-r}-(\mu+1)}\right]$$

$$= \operatorname{Re} h_i(z)\left[\frac{(1+r)[(1+r)-(\mu+1)(1-r)]+2r}{(1+r)[(1+r)-(\mu+1)(1-r)]}\right]. \tag{3.9}$$

The right hand side of (3.9) is positive for $r \geq r_0$. Consequently, $f(z) \in Q_k^{\mu+1}(a,c,\lambda,\alpha)$ for $|z| < r_0$, where $r_0$ is given by (3.5).

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