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APPLICATION OF THE $\left(\frac{G'}{G}\right)$-EXPANSION METHOD FOR THE BURGERS, FISHER AND BURGERS-FISHER EQUATIONS

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Abstract. In this paper, the $(\frac{G'}{G})$-expansion method is used to solve the Burgers, Fisher and Burgers-Fisher equations. New traveling wave solutions are obtained for these equations. It is illustrated that our solutions are more general. It is shown that the proposed method is direct and effective.

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1. Introduction

Most phenomena in real world are described through nonlinear equations. In the recent decades, many effective methods for obtaining exact solutions of nonlinear evolution equations (NLEEs) have been presented, such as Painleve method [17], Jacobi elliptic function method [10], Hirota's bilinear method [7], the sine-cosine function method [13], the tanh-coth function method [4], the exp-function method [6], the homogeneous balance method [12] and so on. Recently, Wang et al. [11] proposed the $(\frac{G'}{G})$-expansion method to find traveling wave solutions of NLEEs. Next, this method was applied to obtain traveling wave solutions of some NLEEs [1,8]. Zhang generalized $(\frac{G'}{G})$-expansion method [18-20]. More recently, this method were proposed to improve and extend Wang et al.'s work [11] to solve variable coefficient equations and high dimensional equations [13-15].

In this work, we apply the $(\frac{G'}{G})$-expansion method to solve the Burgers [2, 9], Fisher [15-16] and Burgers-Fisher [14, 16] equations. The Burgers equation appears in various areas of applied mathematics, such as modeling of fluid dynamics, turbulence, boundary layer behavior, shock wave formation, and traffic flow. The Fisher equation is evolution equation that describes the propagation of a virile mutant in an infinitely long habitat [5]. It also represents a model equation for the evolution of a neutron population in a nuclear reactor [3] and a prototype model for a spreading flame.

2.Description of the $\left(\dfrac{G'}{G}\right)$-expansion method

We suppose that the given nonlinear partial differential equation for $u(x,t)$ to be in the form

$$P(u, u_x, u_t, u_{xx}, u_{xt}, u_{tt}, \ldots) = 0, \tag{1}$$

where P is a polynomial in its arguments. The essence of the $(\frac{G'}{G})$-expansion method can be presented in the following steps:
step 1. Seek traveling wave solutions of Eq. (1) by taking $u(x,t) = U(\xi)$, $\xi = x - ct$, and transform Eq. (1) to the ordinary differential equation

$$Q(U, U', U'', \ldots) = 0, \tag{2}$$

where prime denotes the derivative with respect to ξ.
step 2. If possible, integrate Eq. (2) term by term one or more times. This yields constant(s) of integration. For simplicity, the integration constant(s) can be set to zero.
step 3. Introduce the solution $U(\xi)$ of Eq. (2) in the finite series form

$$U(\xi) = \sum_{i=0}^{N} a_i \left(\frac{G'(\xi)}{G(\xi)}\right)^i, \tag{3}$$

where a_i are real constants with $a_N \neq 0$ to be determined, N is a positive integer to be determined. The function $G(\xi)$ is the solution of the auxiliary linear ordinary differential equation

$$G''(\xi) + \lambda G'(\xi) + \mu G(\xi) = 0, \tag{4}$$

where λ and μ are real constants to be determined.
step 4. Determine N. This, usually, can be accomplished by balancing the linear term(s) of highest order with the highest order nonlinear term(s) in Eq. (2).

step 5. Substituting (3) together with (4) into Eq. (2) yields an algebraic equation involving powers of $(\frac{G'}{G})$. Equating the coefficients of each power of $(\frac{G'}{G})$ to zero gives a system of algebraic equations for a_i, λ, μ and c. Then, we solve the system with the aid of a computer algebra system (CAS), such as Maple, to determine these constants. On the other hand, depending on the sign of the discriminant $\Delta = \lambda^2 - 4\mu$, the solutions of Eq. (4) are well known for us. So, we can obtain exact solutions of the given Eq. (1).

3. Applications

In this section, we apply the $(\frac{G'}{G})$-expansion method to solve the Burgers, Fisher and Burgers-Fisher equations.

3.1 The Burgers equation

The Burgers equation is presented as

$$u_t + uu_x = u_{xx}. \tag{5}$$

We make the transformation $u(x,t) = U(\xi)$, $\xi = x - ct$, where c is the wave speed. Then we get

$$-cU' + UU' - U'' = 0, \tag{6}$$

where prime denotes the derivative with respect to ξ. By once time integrating with respect to ξ, Eq. (6) becomes

$$-cU + \frac{1}{2}U^2 - U' = 0. \tag{7}$$

Balancing U' with U^2 gives $N = 1$. Therefore, we can write the solution of Eq. (7) in the form

$$U(\xi) = a_0 + a_1\Big(\frac{G'}{G}\Big), \quad a_1 \neq 0. \tag{8}$$

By (4) and (8) we derive

$$U^2(\xi) = a_1^2\Big(\frac{G'}{G}\Big)^2 + 2a_0a_1\Big(\frac{G'}{G}\Big) + a_0^2, \tag{9}$$

$$U'(\xi) = -a_1\Big(\frac{G'}{G}\Big)^2 - a_1\lambda\Big(\frac{G'}{G}\Big) - a_1\mu. \tag{10}$$

Substituting Eqs. (8)-(10) into Eq. (7), equating the coefficients of $(\frac{G'}{G})^i (i = 0, 1, 2)$ to zero, we obtain a system of algebraic equations for a_0, a_1, c, λ and μ as follows:

$$\begin{aligned}
\Big(\frac{G'}{G}\Big)^0 &: \quad -ca_0 + \frac{1}{2}a_0^2 + a_1\mu = 0,\\
\Big(\frac{G'}{G}\Big)^1 &: \quad -ca_1 + a_0a_1 + a_1\lambda = 0,\\
\Big(\frac{G'}{G}\Big)^2 &: \quad \frac{1}{2}a_1^2 + a_1 = 0.
\end{aligned} \tag{11}$$

Solving this system by Maple gives

$$a_0 = -\lambda \pm \sqrt{\lambda^2 - 4\mu}, \quad a_1 = -2, \quad c = \pm\sqrt{\lambda^2 - 4\mu}. \tag{12}$$

Substituting the solution set (12) and the corresponding solutions of (4) into (8), we have the solutions of Eq. (7) as follows:
When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function traveling wave solutions

$$U_1(\xi) = \sqrt{\lambda^2 - 4\mu}\left(\pm 1 - \left(\frac{C_1 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}{C_1 \cosh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi + C_2 \sinh \frac{1}{2}\sqrt{\lambda^2 - 4\mu}\xi}\right)\right), \quad (13)$$

where $\xi = x \mp \sqrt{\lambda^2 - 4\mu}t$.
When $\lambda^2 - 4\mu = 0$, we obtain the rational function solutions

$$U_2(\xi) = -\frac{2C_2}{C_1 + C_2 x}, \quad (14)$$

where $\xi = x$.
When $\lambda^2 - 4\mu < 0$, we obtain the trigonometric function traveling wave solutions

$$U_3(\xi) = \pm\sqrt{\lambda^2 - 4\mu} - \sqrt{4\mu - \lambda^2}\left(\frac{-C_1 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}{C_1 \cos \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi + C_2 \sin \frac{1}{2}\sqrt{4\mu - \lambda^2}\xi}\right), (15)$$

where $\xi = x \mp \sqrt{\lambda^2 - 4\mu}t$.

In solutions $U_i(\xi)$ $(i = 1, 2, 3)$, C_1 and C_2 are left as free parameters. It is obvious that hyperbolic, rational and periodic solutions were obtained using the $\left(\frac{G'}{G}\right)$-expansion method, whereas only hyperbolic solutions were obtained in [15-16].

In particular, if we take $C_1 \neq 0$ and $C_2 = 0$, then U_1 becomes

$$U_1(\xi) = c\Big(1 - \tanh\left(\frac{c}{2}\xi\right)\Big), \quad (16)$$

and if we take $C_1 = 0$ and $C_2 \neq 0$, then U_1 becomes

$$U_1(\xi) = c\Big(1 - \coth\left(\frac{c}{2}\xi\right)\Big), \quad (17)$$

where $c = \pm\sqrt{\lambda^2 - 4\mu}$.
The solutions (16) and (17) are the same as Eqs. (15) and (16) in [15-16] respectively. So our solutions logically contains the solutions in [15-16].

3.2 The Fisher equation

Now we consider the Fisher equation

$$u_t = u_{xx} + u(1 - u). \quad (18)$$

We make the transformation $u(x,t) = U(\xi)$, $\xi = x - ct$, where c is the wave speed. Then we get

$$U'' + cU' - U^2 + U = 0, \tag{19}$$

where prime denotes the derivative with respect to ξ. Balancing U'' with U^2 gives $N = 2$. Therefore, we can write the solution of Eq. (19) in the form

$$U(\xi) = a_0 + a_1\left(\frac{G'}{G}\right) + a_2\left(\frac{G'}{G}\right)^2, \quad a_2 \neq 0. \tag{20}$$

Using (4) and (20) we have

$$U'(\xi) = -2a_2\left(\frac{G'}{G}\right)^3 - (a_1 + 2a_2\lambda)\left(\frac{G'}{G}\right)^2 - (a_1\lambda + 2a_2\mu)\left(\frac{G'}{G}\right) - a_1\mu, \tag{21}$$

$$\begin{aligned} U''(\xi) &= 6a_2\left(\frac{G'}{G}\right)^4 + (2a_1 + 10a_2\lambda)\left(\frac{G'}{G}\right)^3 + (3a_1\lambda + 4a_2\lambda^2 + 8a_2\mu)\left(\frac{G'}{G}\right)^2 \\ &+ (a_1\lambda^2 + 2a_1\mu + 6a_2\lambda\mu)\left(\frac{G'}{G}\right) + a_1\lambda\mu + 2a_2\mu^2, \end{aligned} \tag{22}$$

$$\begin{aligned} U^2(\xi) &= a_2^2\left(\frac{G'}{G}\right)^4 + 2a_1a_2\left(\frac{G'}{G}\right)^3 + (a_1^2 + 2a_0a_2)\left(\frac{G'}{G}\right)^2 \\ &+ 2a_0a_1\left(\frac{G'}{G}\right) + a_0^2. \end{aligned} \tag{23}$$

Substituting (20)-(23) into (19), equating coefficients of $\left(\frac{G'}{G}\right)^i (i = 0, 1, 2, 3)$ to zero, we obtain a system of nonlinear algebraic equations a_0, a_1, c, λ and μ as follows:

$$\begin{aligned} \left(\frac{G'}{G}\right)^0 &: a_0 - a_0^2 + a_1\lambda\mu + 2a_2\mu^2 - ca_1\mu = 0, \\ \left(\frac{G'}{G}\right)^1 &: a_1 + a_1\lambda^2 + 2a_1\mu - 2a_1a_0 + 6a_2\lambda\mu - ca_1\lambda - 2ca_2\mu = 0, \\ \left(\frac{G'}{G}\right)^2 &: a_2 + 3a_1\lambda + 4a_2\lambda^2 - a_1^2 + 8a_2\mu - 2a_2a_0 - ca_1 - 2ca_2\lambda = 0, \\ \left(\frac{G'}{G}\right)^3 &: 2a_1 + 10a_2\lambda - 2a_1a_2 - 2ca_2 = 0, \\ \left(\frac{G'}{G}\right)^4 &: 6a_2 - a_2^2 = 0. \end{aligned} \tag{24}$$

Solving this system by Maple gives

$$\begin{aligned}(a) \quad &: \quad a_0 = \frac{1}{4} \mp \frac{\sqrt{6}}{2}\lambda + \frac{3}{2}\lambda^2, \;\; a_1 = \mp\sqrt{6} + 6\lambda, \;\; a_2 = 6, \\ & \quad\; c = \pm\frac{5}{6}(\sqrt{6}), \;\; \mu = \frac{1}{4}\lambda^2 - \frac{1}{24}. \end{aligned} \tag{25}$$

$$\begin{aligned}(b) \quad &: \quad a_0 = \frac{3}{4} \mp i\frac{\sqrt{6}}{2}\lambda + \frac{3}{2}\lambda^2, \;\; a_1 = \mp i\sqrt{6} + 6\lambda, \;\; a_2 = 6, \\ & \quad\; c = \pm i\frac{5}{6}(\sqrt{6}), \;\; \mu = \frac{1}{4}\lambda^2 + \frac{1}{24}. \end{aligned} \tag{26}$$

$$(c) \quad : \quad a_0 = -\frac{1}{2} + \frac{3}{2}\lambda^2, \quad a_1 = 6\lambda, \quad a_2 = 6 \quad c = 0, \quad \mu = \frac{1}{4}(\lambda^2 - 1). \tag{27}$$

$$(d) \quad : \quad a_0 = \frac{3}{2} + \frac{3}{2}\lambda^2, \quad a_1 = 6\lambda, \quad a_2 = 6 \quad c = 0, \quad \mu = \frac{1}{4}(\lambda^2 + 1). \tag{28}$$

Substituting the solutions set (25)-(28) into (20) and using (4), we have the solutions of Eq. (19) as follows:
When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function traveling wave solutions

$$U_1(\xi) = \frac{1}{4} \mp \frac{1}{2}\left(\frac{C_1 \sinh\frac{1}{2\sqrt{6}}\xi + C_2 \cosh\frac{1}{2\sqrt{6}}\xi}{C_1 \cosh\frac{1}{2\sqrt{6}}\xi + C_2 \sinh\frac{1}{2\sqrt{6}}\xi}\right) + \frac{1}{4}\left(\frac{C_1 \sinh\frac{1}{2\sqrt{6}}\xi + C_2 \cosh\frac{1}{2\sqrt{6}}\xi}{C_1 \cosh\frac{1}{2\sqrt{6}}\xi + C_2 \sinh\frac{1}{2\sqrt{6}}\xi}\right)^2, \tag{29}$$

where $\xi = x \pm \frac{5\sqrt{6}}{6}t$ and

$$U_2(\xi) = \frac{-1}{2} + \frac{3}{2}\left(\frac{C_1 \sinh\frac{1}{2}\xi + C_2 \cosh\frac{1}{2}\xi}{C_1 \cosh\frac{1}{2}\xi + C_2 \sinh\frac{1}{2}\xi}\right)^2, \tag{30}$$

where $\xi = x$.
When $\lambda^2 - 4\mu = 0$, by considering Eqs. (25)-(28) this case is impossible.

When $\lambda^2 - 4\mu < 0$, we obtain the periodic function solutions,

$$U_3(\xi) = \frac{3}{4} \mp \frac{i}{2}\left(\frac{-C_1 \sin\frac{1}{2\sqrt{6}}\xi + C_2 \cos\frac{1}{2\sqrt{6}}\xi}{C_1 \cos\frac{1}{2\sqrt{6}}\xi + C_2 \sin\frac{1}{2\sqrt{6}}\xi}\right) + \frac{1}{4}\left(\frac{-C_1 \sin\frac{1}{2\sqrt{6}}\xi + C_2 \cos\frac{1}{2\sqrt{6}}\xi}{C_1 \cos\frac{1}{2\sqrt{6}}\xi + C_2 \sin\frac{1}{2\sqrt{6}}\xi}\right)^2, \tag{31}$$

where $\xi = x \pm \frac{5\sqrt{6}i}{6}t$ and

$$U_4(\xi) = \frac{3}{2} + \frac{3}{2}\left(\frac{-C_1 \sin\frac{1}{2}\xi + C_2 \cos\frac{1}{2}\xi}{C_1 \cos\frac{1}{2}\xi + C_2 \sin\frac{1}{2}\xi}\right)^2, \tag{32}$$

where $\xi = x$.

In solutions $U_i(\xi)$ $(i = 1...4)$, C_1 and C_2 are left as free parameters. It is obvious that hyperbolic and periodic solutions were obtained using the $\left(\frac{G'}{G}\right)$-expansion method, whereas only hyperbolic solutions were obtained in [15-16].

In particular, if we take $C_1 \neq 0$, $C_2 = 0$ and $c = \frac{5}{\sqrt{6}}$ then U_1 becomes

$$U_1(\xi) = \frac{1}{4}\left(1 - \tanh\left(\frac{1}{2\sqrt{6}}\xi\right)\right)^2, \tag{33}$$

and if we take $C_1 = 0$, $C_2 \neq 0$ and $c = \frac{5}{\sqrt{6}}$ then U_1 becomes

$$U_1(\xi) = \frac{1}{4}\left(1 - \coth\left(\frac{1}{2\sqrt{6}}\xi\right)\right)^2. \tag{34}$$

The solutions (33) and (34) are the same as Eqs. (23) and (24) in [16] respectively. It is clear that the solutions of [16] are especial case of our solutions.

3.3 The Burgers-Fisher equation

Now, let us consider the Burgers-Fisher equation

$$u_t = u_{xx} + uu_x + u(1 - u). \tag{35}$$

We make the transformation $u(x, t) = U(\xi)$, $\xi = x - ct$, where c is the wave speed. Then we get

$$cU' + UU' + U'' + U - U^2 = 0, \tag{36}$$

where prime denotes the derivative with respect to ξ. Balancing U'' with UU' gives $N = 1$. Therefore, we can write the solution of Eq. (36) in the form

$$U(\xi) = a_0 + a_1\left(\frac{G'}{G}\right), \quad a_1 \neq 0. \tag{37}$$

Using Eq. (4) and Eq. (36) we have

$$U'(\xi) = -a_1\Big(\frac{G'}{G}\Big)^2 - a_1\lambda\Big(\frac{G'}{G}\Big) - a_1\mu, \tag{38}$$

$$U''(\xi) = 2a_1\Big(\frac{G'}{G}\Big)^3 + 3a_1\lambda\Big(\frac{G'}{G}\Big)^2 + (a_1\lambda^2 + 2a_1\mu)\Big(\frac{G'}{G}\Big) + a_1\lambda\mu, \tag{39}$$

$$U^2(\xi) = a_1^2\Big(\frac{G'}{G}\Big)^2 + 2a_0a_1\Big(\frac{G'}{G}\Big) + a_0^2, \tag{40}$$

$$\begin{aligned} U(\xi)U'(\xi) &= -a_1^2\Big(\frac{G'}{G}\Big)^3 - (a_1^2\lambda + a_0a_1)\Big(\frac{G'}{G}\Big)^2 \\ &\quad -(a_0a_1\lambda + a_1^2\mu)\Big(\frac{G'}{G}\Big) - a_0a_1\mu. \end{aligned} \tag{41}$$

Substituting (37)-(41) into (36), equating coefficients of $\left(\frac{G'}{G}\right)^i (i = 0, 1, 2, 3)$ to zero, we obtain a system of nonlinear algebraic equations a_0, a_1, c, λ and μ as follows:

$$\begin{aligned} \Big(\frac{G'}{G}\Big)^0 &: \ a_1\lambda\mu - ca_1\mu - a_0a_1\mu - a_0^2 + a_0 = 0, \\ \Big(\frac{G'}{G}\Big)^1 &: \ a_1\lambda^2 + 2a_1\mu - 2a_0a_1 - a_1^2\mu - a_0a_1\lambda + a_1 - ca_1\lambda = 0, \\ \Big(\frac{G'}{G}\Big)^2 &: \ 3a_1\lambda - a_1^2 - ca_1 - a_0a_1 - a_1^2\lambda = 0, \\ \Big(\frac{G'}{G}\Big)^3 &: \ 2a_1 - a_1^2 = 0. \end{aligned} \tag{42}$$

Solving this system by Maple gives

$$a_0 = \frac{1}{2} + \lambda, \quad a_1 = 2, \quad c = -\frac{5}{2}, \quad \mu = \frac{1}{4}\lambda^2 - \frac{1}{16}. \tag{43}$$

Substituting the solutions set (43) and the corresponding solutions of (4) into (37), we have the solutions of Eq. (36) as follows:
When $\lambda^2 - 4\mu > 0$, we obtain the hyperbolic function traveling wave solutions

$$U_1(\xi) = \frac{1}{2} + \frac{1}{2}\frac{C_1 \sinh \frac{1}{4}\xi + C_2 \cosh \frac{1}{4}\xi}{C_1 \cosh \frac{1}{4}\xi + C_2 \sinh \frac{1}{4}\xi}, \tag{44}$$

where $\xi = x + \frac{5}{2}t$.
By Eqs. (43) the cases $\lambda^2 - 4\mu < 0$ and $\lambda^2 - 4\mu = 0$ are impossible.
In particular, if we take $C_1 \neq 0$ and $C_2 = 0$, then we have

$$U_1(\xi) = \frac{1}{2}\Big(1 + \tanh\Big(\frac{1}{4}\xi\Big)\Big). \tag{45}$$

If we take $C_1 = 0$ and $C_2 \neq 0$, then we get.

$$U_1(\xi) = \frac{1}{2}\Big(1 + \coth\Big(\frac{1}{4}\xi\Big)\Big). \tag{46}$$

The solutions (45) and (46) are the same as Eqs. (33) and (34) in [16] respectively. Therefore the solutions in [16] are special case of our solutions.

4. Conclusions

In this paper, an implementation of the $(\frac{G'}{G})$-expansion method is given by applying it to three nonlinear equations to illustrate the validity and advantages of the method. As a result, hyperbolic, rational and periodic function solutions with parameters are obtained. The obtained solutions with free parameters may be important to explain some physical phenomena. In this paper it is shown that the obtained solutions are more general.

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