SHARP FUNCTION ESTIMATE FOR MULTILINEAR COMMUTATOR OF SINGULAR INTEGRAL WITH VARIABLE CALDERÓN-ZYGMUND KERNEL

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Abstract: In this paper, we prove the sharp function inequality for the multilinear commutator related to the singular integral operator with variable Calderón-Zygmund kernel. By using the sharp inequality, we obtain the L^p-norm inequality for the multilinear commutator.

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1. Introduction

As the development of singular integral operators, their commutators have been well studied (see [1-4]). Let T be the Calderón-Zygmund singular integral operator, a classical result of Coifman, Rocherberg and Weiss (see [3]) states that commutator $[b, T](f) = T(bf) - bT(f)$(where $b \in BMO(R^n)$) is bounded on $L^p(R^n)$ for $1 < p < \infty$. In [6-8], the sharp estimates for some multilinear commutators of the Calderón-Zygmund singular integral operators are obtained. The main purpose of this paper is to prove the sharp function inequality for the multilinear commutator related to the singular integral operator with variable Calderón-Zygmund kernel. By using the sharp inequality, we obtain the L^p-norm inequality for the multilinear commutator.

2. Notations and Results

First let us introduce some notations (see [4][8][9]). In this paper, Q will denote a cube of R^n with sides parallel to the axes, and for a cube Q let $f_Q = |Q|^{-1}\int_Q f(x)dx$ and the sharp function of f is defined by

$$f^{\#}(x) = \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |f(y) - f_Q|dy.$$

It is well-known that (see [4])

$$f^{\#}(x) \approx \sup_{Q \ni x} \inf_{c \in C} \frac{1}{|Q|} \int_Q |f(y) - C| dy.$$

We say that b belongs to $BMO(R^n)$ if $b^{\#}$ belongs to $L^\infty(R^n)$ and define $||b||_{BMO} = ||b^{\#}||_{L^\infty}$. It has been known that (see [9])

$$||b - b_{2^kQ}||_{BMO} \le Ck||b||_{BMO}.$$

Let M be the Hardy-Littlewood maximal operator, that is that

$$M(f)(x) = \sup_{x \in Q} |Q|^{-1} \int_Q |f(y)| dy;$$

we write that $M_p(f) = (M(|f|^p))^{1/p}$ for $0 < p < \infty$.

For $b_j \in BMO(R^n)(j = 1, \cdots, m)$, set

$$||\vec{b}||_{BMO} = \prod_{j=1}^{m} ||b_j||_{BMO}.$$

Given some functions b_j $(j = 1, \cdots, m)$ and a positive integer m and $1 \le j \le m$, we denote by C_j^m the family of all finite subsets $\sigma = \{\sigma(1), \cdots, \sigma(j)\}$ of $\{1, \cdots, m\}$ of j different elements. For $\sigma \in C_j^m$, set $\sigma^c = \{1, \cdots, m\} \setminus \sigma$. For $\vec{b} = (b_1, \cdots, b_m)$ and $\sigma = \{\sigma(1), \cdots, \sigma(j)\} \in C_j^m$, set $\vec{b}_\sigma = (b_{\sigma(1)}, \cdots, b_{\sigma(j)})$, $b_\sigma = b_{\sigma(1)} \cdots b_{\sigma(j)}$ and $||\vec{b}_\sigma||_{BMO} = ||b_{\sigma(1)}||_{BMO} \cdots ||b_{\sigma(j)}||_{BMO}$.

In this paper, we will study some multilinear commutators as follows.

Definition 1. *Let $K(x) = \Omega(x)/|x|^n \ : \ R^n \setminus \{0\} \to R$. K is said to be a Calderón-Zygmund kernel if*
(a) $\Omega \in C^\infty(R^n \setminus \{0\})$;
(b) Ω is homogeneous of degree zero;
(c) $\int_\Sigma \Omega(x) x^\alpha d\sigma(x) = 0$ for all multi-indices $\alpha \in (N \cup \{0\})^n$ with $|\alpha| = N$, where $\Sigma = \{x \in R^n \ : \ |x| = 1\}$ is the unit sphere of R^n.

Definition 2. *Let $K(x, y) = \Omega(x, y)/|y|^n \ : \ R^n \times (R^n \setminus \{0\}) \to R$. K is said to be a variable Calderón-Zygmund kernel if*
(d) $K(x, \cdot)$ is a Calderón-Zygmund kernel for a.e. $x \in R^n$;
(e) $max_{|\gamma| \le 2n} \left|\left| \frac{\partial^{|\gamma|}}{\partial^\gamma y} \Omega(x, y) \right|\right|_{L^\infty(R^n \times \Sigma)} = M < \infty$.

Suppose b_j $(j = 1, \cdots, m)$ are the fixed locally integrable functions on R^n. Let T be the singular integral operator with variable Calderón-Zygmund kernel as

$$T(f)(x) = \int_{R^n} K(x, x-y) f(y) d(y),$$

where $K(x, x-y) = \dfrac{\Omega(x, x-y)}{|x-y|^n}$ and that $\Omega(x, y)/|y|^n$ is a variable Calderón-Zygmund kernel. The multilinear commutator of singular integral with variable Calderón-Zygmund kernel is defined by

$$T_{\vec{b}}(f)(x) = \int_{R^n} \prod_{j=1}^{m} (b_j(x) - b_j(y)) K(x, x-y) f(y) dy.$$

Note that when $b_1 = \cdots = b_m$, $T_{\vec{b}}$ is just the m order commutator (see [1][5]). It is well known that commutators are of great interest in harmonic analysis and have been widely studied by many authors (see [1-3][5-8]). Our main purpose is to establish the sharp inequality for the multilinear commutator.

Now we state our theorems as following.

Theorem 1. *Let $b_j \in BMO(R^n)$ for $j = 1, \cdots, m$. Then for any $1 < r < \infty$, there exists a constant $C > 0$ such that for any $f \in C_0^\infty(R^n)$ and any $x \in R^n$,*

$$(T_{\vec{b}}(f))^{\#}(x) \le C ||\vec{b}||_{BMO} \left(M_r(f)(x) + \sum_{j=1}^{m} \sum_{\sigma \in C_j^m} M_r(T_{\vec{b}_{\sigma^c}}(f))(x) \right).$$

Theorem 2. *Let $b_j \in BMO(R^n)$ for $j = 1, \cdots, m$. Then $T_{\vec{b}}$ is bounded on $L^p(R^n)$ for $1 < p < \infty$.*

3. Proof of Theorem

To prove the theorems, we need the following lemmas.

Lemma 1. (see[10]) *Let $1 < p < \infty$ and T be the singular integral operator with variable Calderón-Zygmund kernel. Then T is bounded on $L^p(R^n)$.*

Lemma 2. *Let $1 < r < \infty$, $b_j \in BMO(R^n)$ for $j = 1, \cdots, k$. Then*

$$\frac{1}{|Q|} \int_Q \prod_{j=1}^{k} |b_j(y) - (b_j)_Q| dy \le C \prod_{j=1}^{k} ||b_j||_{BMO}$$

and

$$\left(\frac{1}{|Q|}\int_Q \prod_{j=1}^{k}|b_j(y)-(b_j)_Q|^r dy\right)^{1/r} \le C\prod_{j=1}^{k}||b_j||_{BMO}.$$

Proof. Choose $1<p_j<\infty$ $j=1,\cdots,k$ such that $1/p_1+\cdots+1/p_k=1$, we obtain, by Hölder's inequality,

$$\frac{1}{|Q|}\int_Q \prod_{j=1}^{k}|b_j(y)-(b_j)_Q|dy \le \prod_{j=1}^{k}\left(\frac{1}{|Q|}\int_Q |b_j(y)-(b_j)_Q|^{p_j}dy\right)^{1/p_j} \le C\prod_{j=1}^{k}||b_j||_{BMO}$$

and

$$\left(\frac{1}{|Q|}\int_Q \prod_{j=1}^{k}|b_j(y)-(b_j)_Q|^r dy\right)^{1/r} \le \prod_{j=1}^{k}\left(\frac{1}{|Q|}\int_Q |b_j(y)-(b_j)_Q|^{p_j r}dy\right)^{1/p_j r} \le$$

$$C\prod_{j=1}^{k}||b_j||_{BMO}.$$

The lemma follows.

Proof of Theorem 1. It suffices to prove for $f\in C_0^\infty(R^n)$ and some constant C_0, the following inequality holds:

$$\frac{1}{|Q|}\int_Q |T_{\vec{b}}(f)(x)-C_0|dx \le C||\vec{b}||_{BMO}\left(M_r(f)(\tilde{x})+\sum_{j=1}^{m}\sum_{\sigma\in C_j^m}M_r(T_{\vec{b}_{\sigma^c}}(f)(\tilde{x}))\right).$$

Fix a cube $Q=Q(x_0,r)$ and $\tilde{x}\in Q$.

We first consider the **Case m=1** . Write, for $f_1=f\chi_{2Q}$ and $f_2=f\chi_{(2Q)^c}$,

$$T_{b_1}(f)(x)=(b_1(x)-(b_1)_{2Q})T(f)(x)-T((b_1-(b_1)_{2Q})f_1)(x)-T((b_1-(b_1)_{2Q})f_2)(x).$$

Let $C_0=T(((b_1)_{2Q}-b_1)f_2)(x_0)$, then

$$\begin{aligned}
&|T_{b_1}(f)(x)-C_0|\\
\le\ &|(b_1(x)-(b_1)_{2Q})T(f)(x)+T((b_1)_{2Q}-(b_1)f_1)(x)\\
&+T(((b_1)_{2Q}-b_1)f_2)(x)-T(((b_1)_{2Q}-b_1)f_2)(x_0)|\\
\le\ &|(b_1(x)-(b_1)_{2Q})T(f)(x)|+|T((b_1)_{2Q}-(b_1)f_1)(x)|\\
&+|T(((b_1)_{2Q}-b_1)f_2)(x)-T(((b_1)_{2Q}-b_1)f_2)(x_0)|\\
=\ &A(x)+B(x)+C(x).
\end{aligned}$$

For $A(x)$, by Hölder's inequality with exponent $1/r + 1/r' = 1$, we get

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q A(x)dx \\
= &\frac{1}{|Q|}\int_Q |b_1(x)-(b_1)_{2Q}||T(f)(x)|dx \\
\leq &C\left(\frac{1}{|2Q|}\int_{2Q}|b_1(x)-(b_1)_{2Q}|^{r'}dx\right)^{1/r'}\left(\frac{1}{|Q|}\int_Q |T(f)(x)|^r dx\right)^{1/r} \\
\leq &C||b_1||_{BMO}M_r(T(f))(\tilde{x}).
\end{aligned}$$

For $B(x)$, choose p such that $1 < p < r$, and $1 < q < \infty$, $pq = r$, by the boundedness of T on $L^p(R^n)$ and the Hölder's inequality, we obtain

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q B(x)dx \\
= &\frac{1}{|Q|}\int_Q T((b_1-(b_1)_{2Q})f_1)(x)dx \\
\leq &\left(\frac{1}{|Q|}\int_{R^n}[T((b_1-(b_1)_{2Q})f\chi_{2Q})(x)]^p dx\right)^{1/p} \\
\leq &C\left(\frac{1}{|Q|}\int_{R^n}(|b_1(x)-(b_1)_{2Q}||f(x)\chi_{2Q}(x)|)^p dx\right)^{1/p} \\
\leq &C\frac{1}{|Q|^{1/p}}\left(\int_{2Q}|b_1(x)-(b_1)_{2Q}|^{1/pq'}dx\right)^{1/pq'}\left(\int_{2Q}|f(x)|^{pq}dx\right)^{1/pq} \\
\leq &C\left(\frac{1}{|2Q|}\int_{2Q}|f(x)|^r dx\right)^{1/r}\left(\frac{1}{|2Q|}\int_{2Q}|b_1(x)-(b_1)_{2Q}|^{pq'}dx\right)^{1/pq'} \\
\leq &C||b_1||_{BMO}M_r(f)(\tilde{x}).
\end{aligned}$$

For $C(x)$, by [11], we know that

$$T_{\vec{b}}(f)(x) = \sum_{k=1}^{\infty}\sum_{h=1}^{g_k} a_{hk}(x)\int_{R^n}\frac{Y_{hk}(x-y)}{|x-y|^{n+m}}\prod_{j=1}^{m}(b_j(x)-b_j(y))f(y)dy$$

where $g_k \leq Ck^{n-2}$, $||a_{hk}||_{L^\infty} \leq Ck^{-2n}$, $|Y_{hk}(x-y)| \leq Ck^{n/2-1}$ and

$$\left|\frac{Y_{hk}(x-y)}{|x-y|^n} - \frac{Y_{hk}(x_0-y)}{|x_0-y|^n}\right| \leq Ck^{n/2}|x-x_0|/|x_0-y|^{n+1}$$

for $|x-y|>2|x_0-x|>0$. So we get, by Minkowski's inequality and Hölder's inequality,

$$C(x)=\left|\int_{R^n}(K(x,x-y)-K(x_0,x_0-y))((b_1)_{2Q}-b_1(y))f_2(y)dy\right|$$

$$\leq C\int_{(2Q)^c}\left|\frac{\Omega(x,x-y)}{|x-y|^n}-\frac{\Omega(x_0,x_0-y)}{|x_0-y|^n}\right||(b_1)_{2Q}-b_1(y)||f(y)|dy$$

$$\leq C\sum_{l=1}^{\infty}\int_{2^{l+1}Q\setminus 2^lQ}\sum_{k=1}^{\infty}\sum_{h=1}^{g_k}|a_{hk}(x)|\int_{R^n}\left|\frac{Y_{hk}(x-y)}{|x-y|^n}-\frac{Y_{hk}(x_0-y)}{|x_0-y|^n}\right||(b_1)_{2Q}-b_1(y)|\cdot$$

$$\cdot|f(y)|dy\leq C\sum_{k=1}^{\infty}k^{-2n}\cdot k^{n/2}\sum_{l=1}^{\infty}\int_{2^{l+1}Q\setminus 2^lQ}\frac{|x-x_0|}{|x_0-y|^{n+1}}|b_1(y)-(b_1)_{2Q}||f(y)|dy$$

$$\leq C\sum_{k=1}^{\infty}k^{-3n/2}\sum_{l=1}^{\infty}\frac{r}{(2^lr)^{n+1}}\int_{2^{l+1}Q}|b_1(y)-(b_1)_{2Q}||f(y)|dy$$

$$\leq C\sum_{l=1}^{\infty}2^{-l}\left(\frac{1}{|2^{l+1}Q|}\int_{2^{l+1}Q}|b_1(y)-(b_1)_{2Q}|^{r'}dy\right)^{1/r'}\left(\frac{1}{|2^{l+1}Q|}\int_{2^{l+1}Q}|f(y)|^rdy\right)^{1/r}$$

$$\leq C\sum_{l=1}^{\infty}l2^{-l}||b_1||_{BMO}M_r(f)(\tilde{x})\leq C||b_1||_{BMO}M_r(f)(\tilde{x}),$$

thus

$$\frac{1}{|Q|}\int_Q C(x)dx\leq C||b_1||_{BMO}M_r(f)(\tilde{x}).$$

Now, we consider the **Case m $\geq$ 2**, we have known that, for $b=(b_1,\cdots,b_m)$,

$$\begin{aligned}
T_{\vec{b}}(f)(x)&=\int_{R^n}\left[\prod_{j=1}^m(b_j(x)-b_j(y))\right]K(x,x-y)f(y)dy\\
&=\int_{R^n}\prod_{j=1}^m\left[(b_j(x)-(b_j)_{2Q})-(b_j(y)-(b_j)_{2Q})\right]K(x,x-y)f(y)dy\\
&=\sum_{j=0}^m\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma\int_{R^n}(b(y)-(b)_{2Q})_\sigma K(x,x-y)f(y)dy\\
&=(b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})T(f)(x)\\
&\quad+(-1)^mT((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(x)
\end{aligned}$$

$$
\begin{aligned}
&+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma\int_{R^n}(b(y)-b(x))_{\sigma^c}K(x,x-y)f(y)dy\\
=\ &(b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})T(f)(x)\\
&+(-1)^mT((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f)(x)\\
&+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}(-1)^{m-j}(b(x)-(b)_{2Q})_\sigma T_{\vec{b}_{\sigma^c}}(f)(x),
\end{aligned}
$$

thus,

$$
\begin{aligned}
&|T_{\vec{b}}(f)(x)-T((b_1-(b_1)_{2B})\cdots(b_m-(b_m)_{2B})f_2)(x_0)|\\
&\le|(b_1(x)-(b_1)_{2Q})\cdots(b_m(x)-(b_m)_{2Q})T(f)(x)|\\
&+\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}|(b(x)-(b)_{2Q})_\sigma T_{\vec{b}_{\sigma^c}}(f)(x)|\\
&+|T((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_1)(x)|+\\
&|T((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(x)-T((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f_2)(x_0)|\\
&=I_1(x)+I_2(x)+I_3(x)+I_4(x).
\end{aligned}
$$

For $I_1(x)$, by Hölder's inequality with exponent $1/p_1+\cdots+1/p_m+1/r=1$, where $1<p_j<\infty,\ j=1,\cdots,m$, we get

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q I_1(x)dx\\
\le\ &\frac{1}{|Q|}\int_Q|b_1(x)-(b_1)_{2Q}|\cdots|b_m(x)-(b_m)_{2Q}||T(f)(x)|dx\\
\le\ &\prod_{j=1}^m\left(\frac{1}{|Q|}\int_Q|b_j(x)-(b_j)_{2Q}|^{p_j}dx\right)^{1/p_j}\left(\frac{1}{|Q|}\int_Q|T(f)(x)|^rdx\right)^{1/r}\\
\le\ &C||\vec{b}||_{BMO}M_r(T(f))(\tilde{x}).
\end{aligned}
$$

For $I_2(x)$, by the Minkowski's and Hölder's inequality, we get

$$
\begin{aligned}
&\frac{1}{|Q|}\int_Q I_2(x)dx\\
&\le\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\frac{1}{|Q|}\int_Q|(b(x)-(b)_{2Q})_\sigma||T_{\vec{b}_{\sigma^c}}(f)(x)|dx
\end{aligned}
$$

$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}\left(\frac{1}{|2Q|}\int_{2Q}|(b(x)-(b)_{2Q})_\sigma|^{r'}d\mu(x)\right)^{1/r'}\left(\frac{1}{|Q|}\int_Q|T_{\vec{b}_{\sigma^c}}(f)(x)|^r dx\right)^{1/r}$$

$$\leq C\sum_{j=1}^{m-1}\sum_{\sigma\in C_j^m}||\vec{b}_\sigma||_{BMO}M_r(T_{\vec{b}_{\sigma^c}}(f))(\tilde{x}).$$

For $I_3(x)$, choose $1<p<r$, $1<q_j<\infty$, $j=1,\cdots,m$ such that $1/q_1+\cdots+1/q_m+p/r=1$, by the boundedness of T on $L^p(R^n)$ and Hölder's inequality, we get

$$\begin{aligned}
&\frac{1}{|Q|}\int_Q I_3(x)dx\\
\leq\ &\left(\frac{1}{|Q|}\int_{R^n}|T((b_1-(b_1)_{2Q})\cdots(b_m-(b_m)_{2Q})f\chi_{2Q})(x)|^p dx\right)^{1/p}\\
\leq\ &C\left(\frac{1}{|Q|}\int_{R^n}|b_1(x)-(b_1)_{2Q}|^p\cdots|b_m(x)-(b_m)_{2Q}|^p|f(x)\chi_{2Q}(x)|^p dx\right)^{1/p}\\
\leq\ &C\left(\frac{1}{|2Q|}\int_{2Q}|f(x)|^r dx\right)^{1/r}\prod_{j=1}^m\left(\frac{1}{|2Q|}\int_{2Q}|b_j(x)-(b_j)_{2B}|^{pq_j}dx\right)^{1/pq_j}\\
\leq\ &C||\vec{b}||_{BMO}M_r(f)(\tilde{x}).
\end{aligned}$$

For $I_4(x)$, choose $1<p_j<\infty$ $j=1,\cdots,m$ such that $1/p_1+\cdots+1/p_m+1/r=1$, by Minkowski's inequality and Hölder's inequality, we obtain

$$\begin{aligned}
I_4(x)&\leq C\int_{(2Q)^c}\left|\frac{\Omega(x,x-y)}{|x-y|^n}-\frac{\Omega(x_0,x_0-y)}{|x_0-y|^n}\right||\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})||f(y)|dy\\
&\leq C\sum_{l=1}^\infty\int_{2^{l+1}Q\setminus 2^lQ}\left[\sum_{k=1}^\infty\sum_{h=1}^{g_k}|a_{hk}(x)|\int_{R^n}\left|\frac{Y_{hk}(x-y)}{|x-y|^n}-\frac{Y_{hk}(x_0-y)}{|x_0-y|^n}\right|\right]\\
&\qquad\times|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})||f(y)|dy\\
&\leq C\sum_{k=1}^\infty k^{-2n}\cdot k^{n/2}\sum_{l=1}^\infty\int_{2^{l+1}Q\setminus 2^lQ}\frac{|x-x_0|}{|x_0-y|^{n+1}}|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})||f(y)|dy\\
&\leq C\sum_{k=1}^\infty k^{-3n/2}\sum_{l=1}^\infty\frac{r}{(2^lr)^{n+1}}\int_{2^{l+1}Q}|\prod_{j=1}^m(b_j(y)-(b_j)_{2Q})||f(y)|dy
\end{aligned}$$

$$\leq C\sum_{l=1}^{\infty}2^{-l}\left(\frac{1}{|2^{l+1}Q|}\int_{2^{l+1}Q}|f(y)|^{r}dy\right)^{1/r}\cdot$$

$$\cdot\prod_{j=1}^{m}\left(\frac{1}{|2^{l+1}Q|}\int_{2^{l+1}Q}|b_j(y)-(b_j)_{2Q}|^{p_j}dy\right)^{1/p_j}$$

$$\leq C\sum_{l=1}^{\infty}l^m 2^{-l}\prod_{j=1}^{m}||b_j||_{BMO}M_r(f)(\tilde{x})\leq C||\vec{b}||_{BMO}M_r(f)(\tilde{x}),$$

thus

$$\frac{1}{|Q|}\int_Q I_4(x)dx\leq C||\vec{b}||_{BMO}M_r(f)(\tilde{x}).$$

This completes the proof of the theorem.

Proof of Theorem 2. Choose $1<r<p$ in Theorem 1. We first consider the case m=1, we have

$$\begin{aligned}||T_{b_1}(f)||_{L^p} &\leq ||M(T_{b_1}(f))||_{L^p}\leq C\|(T_{b_1}(f))^{\#}\|_{L^p}\\ &\leq C\|M_r(T(f))\|_{L^p}+C\|M_r(f)\|_{L^p}\\ &\leq C\|T(f)\|_{L^p}+C\|M_r(f)\|_{L^p}\\ &\leq C\|f\|_{L^p}+C\|f\|_{L^p}\\ &\leq C\|f\|_{L^p}.\end{aligned}$$

When $m\geq 2$, we may get the conclusion of Theorem 2 by induction. This finishes the proof.

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