MULTIPLICATION OPERATORS ON NON-LOCALLY CONVEX WEIGHTED FUNCTION SPACES

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Abstract. Let X be a completely regular Hausdorff space, E a Hausdorff topological vector space, $CL(E)$ the algebra of continuous operators on E, V a Nachbin family on X and $\mathcal{F} \subseteq CV_b(X, E)$ a topological vector space (for a given topology). If $\pi : X \to CL(E)$ is a mapping and $f \in \mathcal{F}$, let $M_\pi(f)(x) := \pi(x)f(x)$. In this paper we give necessary and sufficient conditions for the induced linear mapping M_π to be a multiplication operator on $\mathcal{F}$ (i.e. a continuous self-mapping on $\mathcal{F}$) in the non-locally convex setting. These results unify and improve several known results.

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1. Introduction

The fundamental work on weighted spaces of continuous scalar-valued functions has been done mainly by Nachbin [12, 13], in the 1960's. Since then it has been studied extensively for a variety of problems by Bierstedt [1,2], Summers [23, 24], Prolla [16, 17], Ruess and Summers [18], Khan [5,6], and many others. The multiplication operators on the Weighted spaces $CV_b(X, E)$ and $CV_0(X, E)$ were first considered by R.K. Singh and J.S. Manhas in [20] in the cases of $\pi : X \to \mathbb{C}$ and $\pi : X \to E$ and later in [21] in the case of $\pi : X \to CL(E)$. In [15], Oubbi gave necessary and sufficient conditions (under some addition assumption) for M_π to be a multiplication operator on a subspace $\mathcal{F}$ of $CV_b(X, E)$. In the above study of multiplication operators, E has been assumed to be a locally convex space ([20, 21, 15]). In this paper we extend some results of the above authors in the general case of E a topological vector space (i.e. not necesserily locally convex). Further, our results include and correct some results of [11] already established for E a TVS.

2. Preliminaries

Henceforth, we shall assume, unless stated otherwise, that X is a completely regular Hausdorff space and E is a non-trivial Hausdorff topological vector space (TVS) with a base $\mathcal{W}$ of closed balanced shrinkable neighbourhoods of 0. (A neighbourhood G of 0 in E is called *shrinkable* [10] if $r\overline{G} \subseteq int\ G$ for $0 \leq r < 1$.) By ([10], Theorems 4 and 5), every Hausdorff TVS has a base of shrinkable neighbourhoods of 0 and also the Minkowski functional ρ_G of any such neighbourhood G is continuous, positively homogeneous and satisfies

$$\overline{G} = \{a \in E : \rho_G(a) \leq 1\},\ int\ G = \{a \in E : \rho_G(a) < 1\}.$$

A *Nachbin family* V on X is a set of non-negative upper semicontinuous function on X, called *weights*, such that given $u, v \in V$ and $t \geq 0$, there exists $w \in V$ with $tu, tv \leq w$ (pointwise) and, for each $x \in X$, ther exists $v \in V$ with $v(x) > 0$; due to this later condition, we sometimes write $V > 0$. Let $C(X, E)$ be the vector space of all continuous E-valued functions on X, and let $C_b(X, E)$ (resp. $C_0(X, E), C_{00}(X, E)$) denote the subspace of $C(X, E)$ consisting of those functions which are bounded (resp. vanish at infinity, have compact support). Further, let

$$\begin{aligned} CV_b(X, E) &= \{f \in C(X, E) : vf(X) \text{ is bounded in } E \text{ for all } v \in V\}, \\ CV_0(X, E) &= \{f \in C(X, E) : vf \text{ vanishes at infinity on } X \text{ for all } v \in V\}. \end{aligned}$$

Clearly, $CV_0(X, E) \subseteq CV_b(X, E)$. When $E = \mathbb{K}(= \mathbb{R} \text{ or } \mathbb{C})$, the above spaces are denoted by $C(X)$, $C_b(X)$, $C_0(X)$, $C_{00}(X)$, $CV_b(X)$, and $CV_0(X)$. We shall denote by $C(X) \otimes E$ the vector subspace of $C(X, E)$ spanned by the set of all functions of the form $\varphi \otimes a$, where $\varphi \in C(X)$, $a \in E$, and $(\varphi \otimes a) = \varphi(x)a$, $x \in X$. The *weighted topology* ω_V on $CV_b(X, E)$ [12, 5, 6] is defined as the linear topology which has a base of neighbourhoods of 0 consisting of all sets of the form

$$N(v, G) = \{f \in CV_b(X, E) : vf(X) \subseteq G\} = \{f \in CV_b(X, E) : \|f\|_{v,G} \leq 1\}$$

where $v \in V$, G is a closed shrinkable set in $\mathcal{W}$, and

$$\|f\|_{v,G} = \sup\{v(x)\rho_G(f(x)) : x \in X\}.$$

The following are some instances of weighted spaces.

(1) If $V = K^+(X) = \{\lambda\chi_X : \lambda > 0\}$, the set of all non-negative constant functions on X, then $CV_b(X,E) = CV_b(X,E)$, $CV_0(X,E) = C_0(X,E)$, and ω_V is the *uniform topology* u.

(2) If $V = S_0^+(X)$, then set of all non-negative upper semi- continuous functions on X which vanish at infinity, then $CV_b(X,E) = CV_0(X,E) = C_b(X,E)$ and ω_V is the *strict topology* β [3,4].

(3) If $V = K_c^+(X) = \{\lambda\chi_K : \lambda > 0$ and $K \subseteq X,\ K$ compact$\}$, then $CV_b(X,E) = CV_0(X,E) = C(X,E)$ and ω_V is the *compact-open topology* k.

(4) If $V = K_f^+(X) = \{\lambda\chi_A : \lambda > 0$ and $A \subseteq X,\ A$ finite$\}$,then $CV_b(X,E) = CV_0(X,E) = C(X,E)$ and ω_V is the *pointwise topology* p.

The assumption $V > 0$ implies thta $p \leq \omega_V$. Recall that $p \leq k$ on $C(X,E)$ and $k \leq \beta \leq u$ on $C_b(X,E)$.

Definition. *For any vector subspace $\mathcal{F} \subseteq C(X,E)$, we define the cozero set of $\mathcal{F}$ by*

$$coz(\mathcal{F}) := \{x \in X : f(x) \neq 0 \text{ for some } f \in \mathcal{F}\}.$$

If $coz(\mathcal{F}) = X$, i.e. if $\mathcal{F}$ does not vanish on X, then $\mathcal{F}$ is said to be essential. In general, $\mathcal{F} = CV_0(X,E)$ and $\mathcal{F} = CV_b(X,E)$ need not be essential.

Definition. (cf. [15]) *(i) A subspace $\mathcal{F}$ of $CV_b(X,E)$ is said to be E-solid if, for every $g \in C(X,E)$, $g \in \mathcal{F} \Leftrightarrow$ for any $G \in \mathcal{W}$, there exist $H \in \mathcal{W},\ f \in \mathcal{F}$ such that*

$$\rho_G \circ g \leq \rho_H \circ f \quad \text{(pointwise) on } coZ(\mathcal{F}). \tag{ES}$$

(ii) A subspace $\mathcal{F}$ of $CV_b(X,E)$ is said to be EV-solid if, for every $g \in CV_b(X,E)$, $g \in \mathcal{F} \Leftrightarrow$ for any $u \in \mathcal{V}, G \in \mathcal{W}$, there exist $u \in \mathcal{V},\ H \in \mathcal{W}$, $f \in \mathcal{F}$ such that

$$v\rho_G \circ g \leq u\rho_H \circ f \quad \text{(pointwise) on } coZ(\mathcal{F})). \tag{EVS}$$

(iii) A subspace $\mathcal{F}$ of $CV_b(X,E)$ is said to have the property (M) if

$$(\rho_G \circ f) \otimes a \in \mathcal{F} \text{ for all } G \in \mathcal{W}, a \in E \text{ and } f \in \mathcal{F}. \tag{M}$$

Note. (1) The classical solid spaces (such as $C_b(\mathbb{R})$ and $C_0(\mathbb{R})$) are nothing but the $\mathbb{K}$-solid ones.

(2) Every EV-solid subspace of $CV_b(X,E)$ is E-solid.

(3) Every E-solid subspace $\mathcal{F}$ of $CV_b(X,E)$ satisfies both conditions (a) $C_b(X)\mathcal{F} \subseteq \mathcal{F}$ and (b) (M).

Examples (i) The spaces $CV_b(X,E)$, $CV_0(X,E)$ and $C_{00}(X,E)$ are all EV-solid.

(ii) $CV_b(X,E) \cap C_b(X,E)$, $CV_0(X,E) \cap C_b(X,E), CV_b(X,E) \cap C_0(X,E)$ and $CV_0(X,E) \cap C_0(X,E)$ are E-solid but need not be EV-solid.

(iii) $C_0(\mathbb{R},\mathbb{C})$ and $C_b(\mathbb{R},\mathbb{C})$ are not $\mathbb{C}V$-solid for $V = \{\lambda e^{-\frac{1}{n}}, n \in \mathbb{N}, \lambda > 0\}$.

Let E and F be TVS, and let $CL(E,F)$ be the set of all continuous linear mappings $T : E \to F$. Then $CL(E,F)$ is a vector space with the usual poinwise operations. If $F = E$, $CL(E) = CL(E,E)$ is an algebra under composition:

$$(ST)(x) = S(T(x)), \quad S,T \in CL(E), \ x \in E,$$

and has identity $I : E \to E$ given by $I(x) = x$ $(x \in E)$.

Definition. *For any collection $\mathcal{A}$ of subsets of E, $CL_{\mathcal{A}}(E,F)$ denotes the subspace of $CL(E,F)$ consisting of those T which are bounded on the members of $\mathcal{A}$ together with the topology $t_{\mathcal{A}}$ of uniform convergence on the elements of $\mathcal{A}$. This topology has a base of neighbourhoods of 0 consisting of all sets of the form*

$$U(D,G) = \{T \in CL_{\mathcal{A}}(E,F) : T(D) \subseteq G\} = \{T \in CL_{\mathcal{A}}(E,F) : \|T\|_{D,G} \leq 1\},$$

where $D \in \mathcal{A}$, G is a closed shrinkable neighbourhood of 0 in F, and

$$\|T\|_{D,G} = \sup\{\rho_G(T(a)) : a \in D\}.$$

If $\mathcal{A}$ consists of all bounded (resp. finite) subsets of E, then we will write $CL_u(E)$ (resp. $CL_p(E)$) for $CL_{\mathcal{A}}(E)$ and t_u (resp. t_p) for $t_{\mathcal{A}}$. Clearly, $t_p \leq t_u$.

For the general theory of topological vector spaces and continuous linear mappings, the reader is refered to [**?**].

Remarks. (1) If $CV_0(X)$ is essential, then clearly $CV_b(X)$ is also essential. The main reason for assuming the essentiality of $CV_0(X)$ in earlier papers [22, 8, 11] as well in the present one is that, for any $x \in X$ and any open neighbourhood U of x in X, we can choose an $f \in CV_0(X)$ with $0 \leq f \leq 1, f(X \backslash U) = 0$, and $f(x) = 1$. This follows from ([13], Lemma 2, p. 69) by taking $E = X, M = CV_0(X) \subseteq C(X), K = \{x\}$, and $U = A_i$ for all $i = 1, ..., n$.

(2) If $V > 0$ and either X is locally compact or $V \subseteq S_0^+(X)$, then $CV_0(X)$ is essential ([22], p. 306). [First suppose that X is locally compact, and let $x \in X$. There exists an $f \in C_{00}(X) \subseteq C_0(X)$ such that $f(x) = 1$. Since $V > 0$, choose $v \in V$ such that $v(x) \neq 0$. Then clearly $vf \in CV_0(X)$ and

$v(x)f(x) \neq 0$. Hence $CV_0(X)$ is essential. Next, suppose $V \subseteq S_0^+(X)$, and let $x \in X$. Choose $v \in V$ such that $v(x) \neq 0$. Since X is completely regular, there exists an $f \in C_b(X)$ such that $f(x) = 1$. Then clearly $vf \in CV_0(X)$ and $v(x)f(x) \neq 0$. Hence $CV_0(X)$ is essential.]

(3) If $CV_0(X)$ is essential and E is a non-trivial TVS, then $CV_0(X) \otimes E$ and hence $CV_b(X) \otimes E$, $CV_b(X, E)$ and $CV_b(X, E)$ are also essential. [In fact, for any $x \in X$, choose φ in $CV_0(X)$ with $\varphi(x) \neq 0$. Then, if a $(\neq 0)$ in E, the function $\varphi \otimes a$ belongs to $CV_0(X) \otimes E$ and clearly $(\varphi \otimes a)(x) = \varphi(x)a \neq 0$. So $CV_0(X) \otimes E$ is essential.]

3. Characterization of Multiplication Operators on $CV_b(X, E)$

In this section, we extend some results of Oubbi [15] to the general TVS setting regarding necessary and sufficient conditions for M_π to be a multiplicative operator on a subspace $\mathcal{F}$ of $CV_b(X, E)$. These results provide, in particular, extension and correction of some results of Singh and Manhas [21], Manhas and Singh [11] and Khan and Thaheem [9].

Definition. *Let $\mathcal{F} \subseteq CV_b(X, E)$ be a topological vector space (for a given topology). Let $\pi : X \to CL(E)$ be a mapping and $F(X, E)$ a set of functions from X into E. For any $x \in X$, we denote $\pi(x) = \pi_x \in CL(E)$, and let $M_\pi : \mathcal{F} \to F(X, E)$ be the linear map defined by*

$$M_\pi(f)(x) := \pi(x)[f(x)] = \pi_x[f(x)], f \in \mathcal{F},\ x \in X.$$

Note that M_π is linear since each π_x is linear. Then M_π is said to be a *multiplication operator* on $\mathcal{F}$ if (i) $M_\pi(\mathcal{F}) \subseteq \mathcal{F}$ and (ii) $M_\pi : \mathcal{F} \to \mathcal{F}$ is continuous on $\mathcal{F}$.

We begin by modifying an example, due to Oubbi [15], in the general setting. This example shows that that $CV_b(X, E)$ may be trivial.

Example 1. Let $X = \mathbb{Q}$, the set of all rationals with the natural topology. This is of course a metrizable space. Consider on X the Nachbin family $V = C^+(X)$ consisting of all non-negative continuous functions. We claim that $CV_b(X, E)$ is reduced to $\{0\}$ for every non-trivial TVS E.

[Indeed, assume that, for a given TVS E, $CV_b(X, E) \neq \{0\}$, and let f $(\neq 0) \in CV_b(X; E)$. Then $f(x_0) \neq 0$ for some $x_0 \in X$. Since E is a Hausdorff TVS, there exists some shrinkable neighbourhood $G \in \mathcal{W}$ so that $\rho_G(f(x_0)) \neq 0$. With no loss of generality, we assume that $\rho_G(f(x_0)) = 1$. Since $\rho_G \circ f$:

$X \to \mathbb{R}$ is contiuous at x_0, taking $\varepsilon = \frac{1}{2}$, there exists $\delta > 0$ such that if $|x - x_0| < \delta$,

$$|\rho_G(f(x)) - \rho_G(f(x_0))| < \frac{1}{2}, \text{ hence } \rho_G(f(x)) > \rho_G(f(x_0)) - \frac{1}{2} = \frac{1}{2}.$$

For an irrational $t \in \mathbb{R}$ with $|t - x_0| < \delta$, the function $v_t : X \to \mathbb{R}^+$ given by

$$v_t(x) = \frac{1}{|t-x|}, x \in X,$$

belongs to $V = C^+(X)$. Now,

$$sup\{v_t(x)\rho_G(f(x)) : x \in X\} \geq v_t(x_0)\rho_G(f(x_0)) = \frac{1}{|t - x_0|} \to \infty \text{ as } t \to x_0,$$

and so $f \notin CV_b(X; E)$, a contradiction.]

Theorem 1. *Let $\pi : X \to CL(E)$ be a map and $\mathcal{F}$ a vector subspace of $CV_b(X, E)$ such that $\mathcal{F}$ is a $C_b(X)$-module and satisfies the condition (M).*

(a) If M_π is a multiplication operator on $\mathcal{F}$, then the following holds: for any $v \in V$ and $G \in \mathcal{W}$, *there exist $u \in V$, $H \in \mathcal{W}$ such that*

$$u(x)a \in H \;\; \textit{implies} \;\; v(x)\pi_x[a] \in G \;\; \textit{for all } x \in coz(\mathcal{F}), \; a \in E;$$

$$\text{i.e.} \quad v(x)\rho_G(\pi_x[a]) \leq u(x)\rho_H(a) \textit{ for all } x \in coz(\mathcal{F}), \; a \in E. \qquad \text{(A)}$$

(b) *Conversely, if, in addition, $\mathcal{F}$ is EV-solid, $M_\pi(\mathcal{F}) \subseteq C(X, E)$ and (A) holds, then M_π is a multiplication operator on $\mathcal{F}$.*

Proof. (a) Suppose M_π is a multiplication operator on $\mathcal{F}$. To prove (A), let $v \in V$ and $G \in \mathcal{W}$. By continuity of $M_\pi : \mathcal{F} \to \mathcal{F}$, $\exists \; u \in V$ and $H \in \mathcal{W}$ such that

$$M_\pi(N(u, H) \cap \mathcal{F}) \subseteq N(v, G) \cap \mathcal{F};$$

i.e., $u(x)f(x) \in H$ implies $v(x)M_\pi(f)(x) \in G$ for all $x \in coz(\mathcal{F})$ and $f \in \mathcal{F}$,

$$\text{i.e.,} \qquad v(x)\rho_G(\pi_x[f(x)]) \leq u(x)\rho_H(f(x)) \text{ for all } x \in coz(\mathcal{F}) \text{ and} f \in \mathcal{F}. \quad (1)$$

In particular, for every $x \in coz(\mathcal{F})$ and $f \in \mathcal{F}$,

$$v(x)\rho_G(\pi_x[f(x)]) \leq u(x)\rho_H(f(x)) \leq \sup_{y \in X} u(y)\rho_H(f(y)). \qquad (2)$$

To verify (A), fix $x_0 \in coz(\mathcal{F})$ and $a \in E$. Choose $g \in \mathcal{F}$ so that $g(x_0) \neq 0$. For each $n \geq 1$, choose $h_n \in C_b(X)$ such that $0 \leq h_n \leq 1$, $h_n(x_0) = 1$, and $h_n = 0$ outside

$$U_n := \{y \in X : u(y) < u(x_0) + \frac{1}{n} \text{ and } \rho_H(g(y)) < 1 + \frac{1}{n}\}.$$

Now, for each $n \geq 1$ and $a \in E$, put $f_{n,a} := h_n \cdot (\rho_H \circ g) \otimes a$. Since $\mathcal{F}$ is a $C_b(X)$-module and satisfies (M), each $f_{n,a} \in \mathcal{F}$. Further, applying (2) to each $f_{n,a}$,

$$v(x_0)\rho_G(\pi_{x_0} f_{n,a}(x_0))) \leq \sup_{y \in X} \ (y)\rho_H(f_{n,a}(y));$$

$$v(x_0)\rho_G(\pi_{x_0}[(h_n \cdot (\rho_H \circ g) \otimes a)(x_0)]) \leq \sup_{y \in X} u(y)\rho_H(h_n \cdot (\rho_H \circ g) \otimes a)(y));$$

$$\text{i.e.,} \quad v(x_0)\rho_G(\pi_{x_0}[h_n(x_0)\rho_H(g(x_0))a]) \leq \sup_{y \in X} u(y)\rho_H(h_n(y)\rho_H(g(y))a). \tag{3}$$

We may assume that $\rho_H(g(x_0)) \neq 0$. [Now either $\rho_H(g(x_0)) = 0$ or $\rho_H(g(x_0)) \neq 0$. If $\rho_H(g(x_0)) = 0$, we change g with another $g_1 \in F$ such that $\rho_H(g_1(x_0)) \neq 0$ as follows: Since $g(x_0) \neq 0$ and E is Hausdorff, there is some $H_1 \in \mathcal{W}$ such that $g(x_0) \notin H_1$. Then choose another $H_2 \in \mathcal{W}$ with $H_2 \subseteq H \cap H_1$. Since $g(x_0) \notin H_1$, clearly $g(x_0) \notin H_2$ and so $\rho_{H_2}(g(x_0)) \geq 1$, hence $\rho_{H_2}(g(x_0)) \neq 0$. Now for some $b \in E$ (e.g. $b \in E \backslash H$) such that $\rho_H(b) \neq 0$, put $g_1 := (\rho_{H_2} \circ g) \otimes b$. By property (M), this is an element of $\mathcal{F}$ and

$$\rho_H(g_1(x_0)) = \rho_H(\rho_{H_2}(g(x_0))b) = \rho_{H_2}((g(x_0))\rho_H(b) \neq 0.]$$

Without loss of generality, we may assume that $\rho_H(g(x_0)) = 1$. Since $h_n(x_0) = 1$, (3) becomes

$$v(x_0) \cdot \rho_G(\pi_{x_0}[a]) \leq \sup_{y \in X} u(y) \cdot h_n(y) \cdot \rho_H(g(y)) \cdot \rho_H(a). \tag{4}$$

We now show that, for $y \in X$,

$$u(y) \cdot h_n(y) \cdot \rho_H(g(y)) \cdot \rho_H(a) \leq (u(x_0) + \frac{1}{n}) \cdot (1 + \frac{1}{n}) \cdot \rho_H(a). \tag{5}$$

Case I: If $y = x_0$, then $h_n(y) = h_n(x_0) = 1, \rho_H(g(y)) = \rho_H(g(x_0)) = 1$, and so

$$u(y) \cdot h_n(y) \cdot \rho_H(g(y)) \cdot \rho_H(a) = u(y) \cdot \rho_H(a).$$

Case II: If $y \neq x_0$ and $y \in U_n$, then $u(y) < u(x_0) + \frac{1}{n}$, $0 \leq h_n(y) \leq 1$,and $\rho_H(g(y)) < 1 + \frac{1}{n}$, and so

$$u(y) \cdot h_n(y) \cdot \rho_H(g(y)) \cdot \rho_H(a) \leq (u(x_0) + \frac{1}{n}) \cdot (1 + \frac{1}{n}) \cdot \rho_H(a).$$

Case III: If $y \neq x_0$ and $y \in X \setminus U_n$, then $h_n(y) = 0$, and so

$$u(y) \cdot h_n(y) \cdot \rho_H(g(y)) \cdot \rho_H(a) = 0 \leq (u(x_0) + \frac{1}{n}) \cdot (1 + \frac{1}{n}) \cdot \rho_H(a).$$

Hence, in each case, (5) holds. Now, by (4) and (5),

$$v(x_0)\rho_G(\pi_{x_0}[a]) \leq (u(x_0) + \frac{1}{n})(1 + \frac{1}{n})\rho_H(a).$$

Since n is arbitrary, $v(x_0)\rho_G(\pi_{x_0}[a]) \leq u(x_0)\rho_H(a)$.

(b) Suppose $\mathcal{F}$ is EV-solid, $M_\pi(\mathcal{F}) \subseteq C(X, E)$ and (A) holds. Then, for every $f \in \mathcal{F}$, $M_\pi(f) \in C(X, E)$, hence $M_\pi(f)$ is continuous on X. So we only need to show that (i) $M_\pi(\mathcal{F}) \subseteq \mathcal{F}$, (ii) $M_\pi : \mathcal{F} \to \mathcal{F}$ is continuous on $\mathcal{F}$.

(i) Let $f \in \mathcal{F}$, and let $v \in V$ and $G \in \mathcal{W}$. By (A), there exist $u \in V$ and $H \in \mathcal{W}$ such that

$$u(x)a \in H \text{ implies that } v(x)\pi_x[a] \in G \text{ for all } x \in coz(\mathcal{F}),\ a \in E.$$

In particular, since $f(x) \in E$,

$$u(x)f(x) \in H \text{ implies that } v(x)\pi_x[f(x)] \in G \text{ for all } x \in coz(\mathcal{F});$$

$$\text{i.e., } v\rho_G \circ M_\pi(f) \leq u\rho_H \circ f \text{ (pointwise) on } coz(\mathcal{F})).$$

Since $\mathcal{F}$ is EV-solid, this implies that $M_\pi(f) \in \mathcal{F}$. Hence $M_\pi(\mathcal{F}) \subseteq \mathcal{F}$.

(ii) To show that $M_\pi : \mathcal{F} \to \mathcal{F}$ is continuous on $\mathcal{F}$, let $v \in V$ and $G \in \mathcal{W}$. Again, by (A), as above, there exist $u \in V$ and $H \in \mathcal{W}$ such that

$$u(x)a \in H \text{ implies } v(x)\pi_x[a] \in G \text{ for all } x \in coz(\mathcal{F}),\ a \in E. \qquad (A_1)$$

We claim that $M_\pi(N(u, H) \cap \mathcal{F}) \subseteq N(v, G) \cap \mathcal{F}$. [Let $f \in N(u, H) \cap \mathcal{F}$. Then $u(x)f(x) \in H$ for all $x \in coz(\mathcal{F})$, and so by (A_1), $v(x)\pi_x[f(x)] \in G$, or that $v(x)M_\pi(f)(x) \in G$ for all $x \in coz(\mathcal{F})$; hence $M_\pi(f) \in N(v, G) \cap \mathcal{F}$.] Thus M_π is continuous on $\mathcal{F}$. $\square$

Corollary 1. *Let $\pi : X \to CL(E)$ be a map. If M_π is a multiplication operator on $CV_b(X, E)$, the so is on any EV-solid subspace $\mathcal{F}$ (e.g. $\mathcal{F} = CV_0(X, E)$ or $C_0(X, E)$) of $CV_b(X, E)$.*

The converse of the above corollary fails to hold in general, even in the scalar case, as the following example shows. We have elaborated this example due to some misprints pointed out to us by the author of [15].

Example 2. ([15], p. 116) Let $X := [0, 1] \cup \mathbb{Q}_{[1,2]}$, where $\mathbb{Q}_{[1,2]}$ denotes the set of all the rationals contained in $[1, 2]$, $E := \mathbb{C}$ and $v_{\sqrt{2}} \in C^+(X)$, where

$$v_{\sqrt{2}}(x) = \frac{1}{|x - \sqrt{2}|}, x \in X.$$

Take $\pi = v_{\sqrt{2}} : X \to CL(\mathbb{C}) = \mathbb{C}$ and $V = K^+(X) = \{\lambda 1 : \lambda \geq 0\}$. We have:

(i) $CV_b(X) = C_b(X)$ with the sup norm.

(ii) Since $CV_0(\mathbb{Q}_{[1,2]}) = \{0\}$, for any $f \in CV_0(X) \subseteq C_b(X)$, $f(\mathbb{Q}_{[1,2]}) = \{0\}$; hence

$$CV_b(X) = CV_0(X) = C_1[0, 1] := \{f \in C[0, 1] : f(1) = 0\}$$

with the uniform norm. This is a Banach algebra.

(iii) It is easy to see that $M_\pi : f \to \pi f$ is a multiplication operator on $\mathcal{F} = CV_0(X)$ but not on $CV_b(X)$. [In fact, in view of Condition (A) of above theorem, for any $\lambda > 0$, there exists some $\mu > 0$ such that

$$\lambda v_{\sqrt{2}}(x)|a| \leq \mu|a| \text{ for all } x \in X, a \in \mathbb{C}; \text{ i.e., } \frac{1}{|x - \sqrt{2}|} \leq \frac{\mu}{\lambda} \text{ for all } x \in X.$$

This clearly holds for all $x \in [0, 1]$ but not for $x \in X$ sufficiently close to $\sqrt{2}$.]
$\square$

Remark. Theorem 3.1 of [11] is an anologue of the above theorem for weighted composition operators in the case of $\mathcal{F} = CV_0(X, E)$ with E a non-locally convex TVS. However, there seems to be a minor error in its proof. In the course of the proof in [11] on p. 279, the authors have obtained $v(x_0)\pi_{x_0}(f(\phi(x_0))) \in G$ by using the inclusion $\frac{1}{2}G + \frac{1}{2}G \subseteq G$, where G is a balanced neighbourhood of 0 in E. But this need not hold unless G is a convex neighbourhood of 0 in E. We can rectify the argument, as follows. Choose earlier in the proof a balanced neighbouhood U of 0 in E with $U + U \subseteq G$. Now simply replace $\frac{1}{2}G$ by U.

Using Theorem 1, we now establish the following result concerning the necessary and sufficient conditions for M_π to be a multiplication operator in the case of $\pi : X \to CL_p(E)$.

Theorem 2. *Let $\mathcal{F}$ be an EV-solid subspace of $CV_b(X,E)$ and $\pi : X \to CL_p(E)$ a continuous function. Suppose that, for every $x \in X$, there exists a neighbourhood D of x with $\pi(D)$ equicontinuous on E. Then M_π is a multiplication operator on $\mathcal{F} \Leftrightarrow$* the *condition* (A) *holds.*

Proof. ($\Rightarrow$) Suppose M_π is a multiplication operator on $\mathcal{F}$. Since $\mathcal{F}$ is EV-solid and, in particular, E-solid, $\mathcal{F}$ is a $C_b(X)$-module and satisfies (M). Hence, by Theorem 1, (A) holds.

($\Leftarrow$) Suppose (A) holds. Since $\mathcal{F}$ is EV-solids, in view of Theorem 1, we only need to show that $M_\pi(\mathcal{F}) \subseteq C(X,E)$. Let $f \in \mathcal{F}$, and let $x_0 \in X$ and $G \in \mathcal{W}$. Choose a balanced $H \in \mathcal{W}$ with $H + H \subseteq G$. By hypothesis, there exists an open neighbourhood D of x_0 such that $\pi(D)$ is equicontinuous on E. So there exists a balanced $H_1 \in \mathcal{W}$ such that

$$\pi_x(H_1) \subseteq H \text{ for all } x \in D. \tag{1}$$

By continuity of f at $x_0 \in X$, there exists an open neighbourhood D_1 of x_0 in X such that

$$f(x) - f(x_0) \in H_1 \text{ for all } x \in D_1. \tag{2}$$

Also, since $\pi : X \to CL_p(E)$ is continuous at x_0 and the set

$$M(\{f(x_0)\}, H) = \{T \in CL(E) : T(f(x_0)) \in H\}$$

is a t_p-neighbourhood of 0 in $CL(E)$, there exists a neighbourhood D_2 of x_0 in X such that

$$\pi(x) - \pi(x_0) \in M(\{f(x_0)\}, H), \text{ or } (\pi_x - \pi_{x_0})[f(x_0)] \in H \text{ for all } x \in D_2. \tag{3}$$

Hence, for any $x \in D \cap D_1 \cap D_2$, using (1)-(3)

$$\begin{aligned} M_\pi(f)(x) - M_\pi(f)(x_0) &= \pi_x[f(x)] - \pi_{x_0}[f(x_0)] \\ &= \pi_x[f(x)] - \pi_x[f(x_0)] + \pi_x[f(x_0)] - \pi_{x_0}[(f(x_0)] \\ &= \pi_x[f(x) - f(x_0)] + (\pi_x - \pi_{x_0})[f(x_0)] \\ &\in \pi_x(H_1) + H \subseteq H + H \subseteq G. \end{aligned}$$

Therefore, $M_\pi(f)$ is continuous at x_0, and so on X. Since $f \in \mathcal{F}$ is arbitrary, $M_\pi(\mathcal{F}) \subseteq C(X,E)$. □

Next, we provide extensions of results of Singh-Manhas [21] to a wider class of completely regular spaces. Following Oubbi [15], we introduce a class of $\gamma_{\mathbb{R}}$-spaces which includes as a special case the $k_{\mathbb{R}}$-spaces and pseudo-compact spaces.

Definition. *Let γ be a property of a net $\{x_\alpha : \alpha \in I\}$ which may satisfy or not. A net $\{x_\alpha : \alpha \in I\}$ is called a γ-net if it possesses a certain property γ. In particular, we shall be interested in the following types of nets:*

Definition. *A net $\{x_\alpha : \alpha \in I\}$ is called:*

(i) an s-net if $\{x_\alpha : \alpha \in I\} = \{x_n : n \in N\}$, a sequence (i.e. if $I = \mathbb{N}$);

(ii) a k-net if $\{x_\alpha : \alpha \in I\}$ is contained in a compact set;

(iii) a b-net if $\{x_\alpha : \alpha \in I\}$ is bounding (i.e. every continuous scalar function on X is bounded on $\{x_\alpha : \alpha \in I\}$).

Definition. *(a) Let X and Y be topological spaces. A function $f : X \to Y$ is said to be γ-continuous if, for every $x \in X$ and every γ-net $\{x_\alpha : \alpha \in I\} \subseteq X$,*

$$x_\alpha \to x \text{ implies } f(x_\alpha) \to f(x) \text{ in } Y.$$

(b) The space X is called a $\gamma_{\mathbb{R}}$-space if every γ-continuous function from X into $\mathbb{R}$ (or equivalently into any completely regular space) is continuous on X.

Examples. (1) The $k_{\mathbb{R}}$-spaces are nothing but the classical ones (as defined above).

(2) Every sequential space is a $s_{\mathbb{R}}$-space.

(3) Every pseudo-compact space is a $b_{\mathbb{R}}$-space.

Clearly, every $s_{\mathbb{R}}$-space is a $k_{\mathbb{R}}$-space and every $k_{\mathbb{R}}$-space is a $b_{\mathbb{R}}$-space.

Definition. *Let V be a Nachbin family on X. A net $\{x_\alpha : \alpha \in I\}$ is called a V-net if there exists some $v \in V$ such that*

$$\{x_\alpha : \alpha \in I\} \subseteq S_{v,1} := \{x \in X : v(x) \geq 1\},$$

i.e. $v(x_\alpha) \geq 1$ for all $\alpha \in I$.

Using V-nets, we hence get V-continuity. In particular, we get the classical $V_{\mathbb{R}}$-spaces intorudced by Bierstedt [2]:

Definition. *X is said to be a $V_{\mathbb{R}}$-space if a function $f : X \to \mathbb{R}$ is continuous whenever, for each $v \in V$, the restriction of f to $S_{v,1}$ is continuous. If $V = K^+(X)$, then X is a $V_{\mathbb{R}}$-space means X is a $k_{\mathbb{R}}$-space. (See also [18], p. 11).*

Definition. *If $\mathcal{A} \subseteq CL(E)$ consists of the γ-nets $\{x_\alpha : \alpha \in I\}$ converging to 0, then we denote $CL_{\mathcal{A}}(E)$ by $CL_\gamma(E)$ and $t_{\mathcal{A}}$ by t_γ. It is then clear that $t_p \leq t_s \leq t_k \leq t_b \leq t_u$.*

Theorem 3. *Let $\mathcal{F}$ be an EV-solid subspace of $CV(X,E)$ and X a $\gamma_{\mathbb{R}}$-space with $\gamma \in \{s,c,k,b\}$ and also γ is conserved by continuous functions (i,e whenever $\{x_\alpha\}$ is a γ-net in X, $\{f(x_\alpha)\}$ is a γ-net in E for all $f \in C(X,E)$) and let $\pi : X \to CL_\gamma(E)$ be a continuous map. Then M_π is a multiplication operator on $F \Leftrightarrow (A)$ holds.*

Proof. ($\Rightarrow$) Suppose M_π is a multiplication operator on $\mathcal{F}$. Since $\mathcal{F}$ is EV-solid, $\mathcal{F}$ is $C_b(X)$-module and satisfies (M). Hence, by Theorem 1, (A) holds.

($\Leftarrow$) Suppose (A) holds. Since $\mathcal{F}$ is EV-solid, by Theorem 1, we only need to show that $M_\pi(\mathcal{F}) \subseteq C(X,E)$. Let $f \in C(X,E)$. Since X is a $\gamma_{\mathbb{R}}$-space, to show that $M_\pi(f)$ is continuous on X, it suffices to show that $M_\pi(f)$ is γ-continuous on X. Let $x_0 \in X$ and $G \in \mathcal{W}$. Choose a balanced $H \in \mathcal{W}$ such that $H + H \subseteq G$. Let $\{x_\alpha : \alpha \in I\}$ be a γ-net in X with $x_\alpha \to x_0$. Since γ is conserved by continuous functions, $\{f(x_\alpha) : \alpha \in I\}$ is also a γ-net with $f(x_\alpha) \to f(x_0)$. Since $\pi : X \to CL_\gamma(E)$ is continuous, there exists a neighbourhood D of x_0 in X such that

$$\pi_y - \pi_{x_0} \in M(\{f(x_\alpha) : \alpha \in I\}, H) \text{ for all } y \in D. \tag{1}$$

Since $x_\alpha \to x_0$, there exists $\alpha_0 \in I$ such that $x_\alpha \in D$ for all $\alpha \geq \alpha_0$. Hence

$$(\pi_y - \pi_{x_0})(f(x_\alpha)) \in H \text{ for all } \alpha \geq \alpha_0. \tag{2}$$

Since π_{x_0} is continuous on E and, in particular, at $0 \in E$, there exists a balanced $H_1 \in \mathcal{W}$ such that

$$\pi_{x_0}(H_1) \subseteq H. \tag{3}$$

Since $f(x_\alpha) \to f(x_0)$, there exists $\alpha_1 \in I$ such that

$$f(x_\alpha) - f(x) \in H_1 \text{ for all } \alpha \geq \alpha_1. \tag{4}$$

Choose $\alpha_2 \in I$ with $\alpha_2 \geq \alpha_0$ and $\alpha_2 \geq \alpha_1$. Then, for $\alpha \geq \alpha_2$, using (1)-(4)

$$\begin{aligned} M_\pi(f)(x_\alpha) - M_\pi(f)(x_0) &= \pi_{x_\alpha}(f(x_\alpha)) - \pi_{x_0}(f(x_0)) \\ &= (\pi_{x_\alpha} - \pi_{x_0})(f(x_\alpha)) + \pi_{x_0}[f(x_\alpha) - f(x_0)] \\ &\in H + \pi_{x_0}[H_1] \subseteq H + H \subseteq G. \end{aligned}$$

Therefore $M_\pi(f)$ is γ-continuous at x_0 and then on the whole of X. $\square$

As a particular, we obtain the following generalization of ([21], Theorem 3).

Corollary 2. *Let $\mathcal{F}$ be a EV-solid subspace of $CV(X,E)$ and $\pi : X \to CL_u(E)$ a continuous map. If X is a $b_{\mathbb{R}}$-space (in particular, $k_{\mathbb{R}}$-space, a sequential space or a pseudo-compact one), then M_π is a multiplication operator on $\mathcal{F} \Leftrightarrow$ (A) holds.*

Proof. Take $\gamma = b$ in Theorem 3. □

In the following result we consider conditions which ensure the continuity of $\pi : X \to CL_p(E)$ in the case of $t_{\mathcal{A}} = t_p$ and thus obtain a kind of converse to Theorems 2 and 3.

Theorem 4. *Let $\mathcal{F}$ be a subspace of $C(X,E)$ satisfying (M) and $\pi : X \to CL_p(E)$ a map. If $M_\pi(\mathcal{F}) \subseteq C(X,E)$, then π is continuous on $coz(\mathcal{F})$.*

Proof. Let $x_0 \in coz(\mathcal{F})$, and let $a \in E$ and $G \in \mathcal{W}$. We show that there exists a neighbourhood D of x_0 in X such that

$$\pi_y - \pi_{x_0} \in N(\{a\}), G), \text{ i.e. } \rho_G[(\pi_y - \pi_{x_0})(a)] \leq 1 \text{ for all } y \in D.$$

Choose balanced $J, S \in \mathcal{W}$ with $J + J \subseteq G$ and $S + S \subseteq J$. Since $x_0 \in coz(\mathcal{F})$ and $coz(\mathcal{F})$ is Hausdorff, there exists an $f \in \mathcal{F}$ and $H \in \mathcal{W}$ such that $f(x_0) \notin H$, and so $\rho_H(f(x_0)) \geq 1$. We may assume that $\rho_H(f(x_0)) = 1$. Put

$$D_1 = \{y \in X : |\rho_H(f(y) - 1| < \frac{1}{2}\}.$$

Clearly, D_1 is open. Further, $D_1 \subseteq coz(\mathcal{F})$ [since, for any $y \in D_1, \frac{1}{2} < \rho_H(f(y)) < 3/2$ and so $\rho_H(f(y)) \neq 0$]. Since $M_\pi(\mathcal{F}) \subseteq C(X,E)$, the map $M_\pi(f) : y \to \pi_y(f(y))$ and in particular $M_\pi(\rho_H \circ f \otimes a) : y \to \pi_y(\rho_H(f(y))a)$ is continuous from X to E (at $y = x_0$), so there exists an open neighbourhood D_2 of x_0 in X such that

$$\rho_S[\pi_y(\rho_H \circ f(y)a) - \pi_{x_0}(\rho_H \circ f(x_0)a)] < \frac{1}{4} \text{ for all } y \in D_2. \tag{1}$$

Case I. Suppose $\rho_S[\pi_{x_0}(a)] = 0$. Then, since $S + S \subseteq J$, (1) gives

$$\begin{aligned} \rho_J[\pi_y(\rho_H(f(y)a)] &\leq \rho_S[\pi_y(\rho_H(f(y)a)) - \pi_{x_0}[\rho_H(f(x_0)a))] + \rho_S[\pi_{x_0}(\rho_H(f(x_0)a))] \\ &< \frac{1}{4} + \rho_H(f(x_0)).0 = \frac{1}{4}. \end{aligned} \tag{2}$$

If also $y \in D_1$ (i.e. $y \in D_1 \cap D_2$), $\rho_H(f(y) > \frac{1}{2}$, so by (2),

$$\rho_J[\pi_y(a)] \leq \frac{1}{\rho_H(f(y))} \cdot \frac{1}{4} < 2 \cdot \frac{1}{4} = \frac{1}{2}. \tag{3}$$

Hence, since $S \subseteq J$ and $J + J \subseteq G$, for any $y \in D_1 \cap D_2$, (3) gives

$$\rho_G[\pi_y(a) - \pi_{x_0}(a))] \leq \rho_J[\pi_y(a)] + \rho_J[\pi_{x_0}(a)] < \frac{1}{2} + \rho_S[\pi_{x_0}(a)) = \frac{1}{2}.$$

This proves the continuity of π at x_0.

Case II. Suppose $\rho_S[\pi_{x_0}(a)] \neq 0$. Put

$$D_3 = \{y \in X : \left| \frac{1}{\rho_H(f(x_0))} - 1 \right| < \frac{1}{\rho_S(\pi_{x_0}(a))}\}.$$

Let $y \in D_1 \cap D_2 \cap D_3$. Since $S + S \subseteq G$,

$$\begin{aligned} & \rho_G[\pi_y(a) - \pi_{x_0}(a)] \leq \\ & \rho_S[\frac{\pi_y[\rho_H(f(y))a]}{\rho_H(f(y))} - \frac{\pi_y[\rho_H(f(x_0))a]}{\rho_H(f(y))}] + \rho_S[\frac{\pi_y(\rho_H(f(x_0))a]}{\rho_H(f(y))} - \pi_{x_0}(a)] \\ \leq \; & \frac{1}{\rho_H(f(y))}\rho_S[\pi_y(\rho_H(f(y))a] - \pi_{x_0}(\rho_H(f(x_0))a)] + \left|\frac{1}{\rho_H(f(y))} - 1\right| \rho_S[\pi_{x_0}(a))] \\ < \; & 2 \cdot \frac{1}{4} + \frac{1}{2\rho_S(\pi_{x_0}(a))} \cdot \rho_S(\pi_{x_0}(a)) = 1. \end{aligned}$$

So, also in this case, π is continuous at x_0. $\square$

Remark. We mention that the above results are obtained for the subset $coz(\mathcal{F})$ of X. Consequently, these results provide correction of corresponding results of [21] and [9] by assuming that the spaces $\mathcal{F} = CV_b(X, E)$ and $\mathcal{F} = CV_0(X, E)$ are essential (i.e. $coz(\mathcal{F}) = X$).

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