VARIOUS REPRESENTATIONS FOR THE SYSTEM OF HYPERBOLIC EQUATIONS WITH SINGULAR COEFFICIENTS

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Abstract. Many new integral representations for the linear system of non-standard hyperbolic equations with two singular lines in infinite regions, are obtained. Furthermore, the obtained results are used to investigate some new boundary value problems in infinite regions. Example of the obtained results is set. The paper is devoted to investigate two different cases.

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1. Introduction

Hyperbolic differential equations with singular coefficients or singular surfaces possess importance in diverse areas of mathematical physics and mathematical engineering, including elasticity, hydrodynamics, thermodynamics and other problems [1-7]. Furthermore, it is also well known that hyperbolic differential equations with one or more singular lines occur in engineering and physical processes. For example, the non-model hyperbolic equation of second order with two singular lines is employed to describe the transformation spectrum of electric signals on long lines with variable parameters in the theory of the electric flail [3-7].

Radjabov N. and co-workers recently have investigated some singular equations. The contributions [3, 4] examin certain classes of singular elliptic and hyperbolic partial differential equations,the contributions [4,5,7] discuss non-model linear hyperbolic equations with singular points or singular surfaces in finite and infinite regions while the contribution [6] discuss non-model linear hyperbolic equations with regular coefficients in infinite regions. The obtained solutions bad been used to solve many boundary value problems. More detailed information for the study of these equations can be found in a number of works [3-7].

2. Main Results

Let D be the following infinite region:

$$D = \{(x, y) : -\infty \leq x \leq \infty, -\infty \leq y \leq \infty\},$$

Which is bounded by

$$\Gamma_1 = \{-\infty \leq x \leq \infty, y = 0\},$$
$$\Gamma_2 = \{x = 0, -\infty \leq y \leq \infty\}.$$

In the region D we consider the following system:

$$xy.\frac{\partial^2 U_s(x,y)}{\partial x \partial y} + \sum_{j=1}^{n} \left[x.a_{js}(x,y).\frac{\partial U_j(x,y)}{\partial x} + y.b_{js}(x,y).\frac{\partial U_j(x,y)}{\partial y} + c_{js}(x,y).U_j(x,y) \right] = \\ = f_s(x,y), 1 \leq s \leq n \tag{1}$$

where the coefficients $a_{js}(x, y), b_{js}(x, y), c_{js}(x, y)$ and $f_s(x, y)$ are given continuous functions.

In the present paper for the system (1) depending on equation coefficients a series of new integral representations are obtained. These integral representations are used for the solution of a number of boundary value problems, the following statements being valid.

Case 1.

Theorem 1 *Let the coefficients in system (1) satisfy the following condition:*

a) $a_{ss}(x, y)$ with respect to the variable y satisfy Holder's conditions and with a variable x have continuous derivatives of the first order.

b) $b_{ss}(x, y)$ of the variable x satisfy Holder's conditions and with a variable y have continuous derivatives of the first order and the function $\frac{\partial b_{ss}(x,y)}{\partial y}$ with respect to the variable x satisfy Holder's conditions.

c) $a_{js}(x, y))(j \neq s)$ of the variable x have continuous derivatives of the first order and continuous with the variable y.

d) $b_{js}(x, y))(j \neq s)$ of the variable y have continuous derivatives of the first order and continuous with the variable x.

e) In a neighborhood of Γ_1 *and* Γ_2 *the functions* $a_{js}(x,y), b_{js}(x,y), c_{js}(x,y)$ *satisfy the following Holder's conditions:*

$$c_s^{(1)}(x,y) = 0(|x|^{\alpha_s}), c_s^{(1)}(x,y) = 0(|y|^{\beta_s}), \alpha_s > 0, \beta_s > 0 \tag{2}$$

(where $c_s^{(1)}(x,y)$ *is the first order set of all continuous functions)*

$$b_{js}(x,y) = 0(|x|^{\alpha_{js}}), \quad \alpha_{js} > 0 \textit{ in a neighborhood of } \Gamma_1, \tag{3}$$

$$b_{js}(x,y) = 0(|y|^{\beta_{js}}), \quad \beta_{js} > 0 \textit{ in a neighborhood of } \Gamma_2, \tag{4}$$

$$\frac{\partial b_{js}(x,y)}{\partial y} = 0(|x|^{\delta_{js}}), \quad \delta_{sj} > 0 \textit{ in a neighborhood of } \Gamma_1, \tag{5}$$

$$a_{js}(x,y) = 0(|x|^{\mu_{js}}), \qquad \mu_{js} > 0 \textit{ in a neighborhood of } \Gamma_2, \tag{6}$$

$$\frac{\partial a_{js}(x,y)}{\partial y} = 0(|x|^{\gamma_{js}}), \quad \gamma_{js} > 0 \textit{ in a neighborhood of } \Gamma_1, \tag{7}$$

f) $0 < a_{ss}(x,0) < 1, \quad 0 < b_{ss}(0,y) < 1.$

Then any solution for the system (1) within the class $C^2(D)$ *can be represented in the form*

$$U_s(x,y) = e^{-w_2^s(x,y)}|y|^{-a_{ss}(x,0)}V_s(x,y), \tag{8}$$

where $V_s(x,y)$ *is a solution of the system (1) Volterra integral equations of the second type in the form:*

$$V_s(x,y) - \int_y^\infty d\tau \int_x^\infty M_{js}^{(1)}(x,y;t,\tau)V_s(t,\tau)dt -$$

$$- \sum_{j=1, j\neq s}^{n} \left\{ \int_y^\infty d\tau \int_x^\infty M_{js}^{(2)}(x,y;t,\tau)V_j(t,\tau)dt + \right.$$

$$\left. \int_x^\infty V_j(t,y).M_{js}^{(3)}(x,y;t)dt + \int_y^\infty M_{js}^{(4)}(x,y;\tau)V_j(x,\tau)d\tau \right\} = E_s^{(1)}(x,y), \tag{9}$$

$$\begin{gathered} E_s^{(1)} = \Phi_s(x) + \int_y^\infty e^{w_2^s(x,\tau)-w_1^s(x,\tau)}|\tau|^{a_{ss}(x,0)-b_{ss}(0,\tau)}.\Psi_s(\tau)d\tau + \\ \int_y^\infty e^{w_2^s(x,\tau)-w_1^s(x,\tau)}|\tau|^{a_{ss}(x,0)-b_{ss}(0,\tau)}.\int_x^\infty e^{w_1^s(t,\tau)}|t|^{b_{ss}(0,\tau)}\frac{f_s(t,\tau)}{t\tau}dt, \end{gathered} \tag{10}$$

where $\Phi_s(x), \Psi_s(y)$ *are given continuous functions of the variables* x *and* y. *Moreover,* $\Phi_s(x) \in C^2(\Gamma_1), \Psi_s(y) \in C^1(\Gamma_2)$, *[*$M_{js}^{(1)}(x,y;t,\tau)$, $M_{js}^{(2)}(x,y;t,\tau)$, $M_{js}^{(3)}(x,y;t)$, $M_{js}^{(4)}(x,y;t,\tau)$ *as given later in pages 8,9], [*$w_1^s(x,y), w_s^s(x,y)$ *as given later in the proof by the formulae (15), (16)].*

Proof. Let the coefficients in system (1) satisfy: $a_{ss}(x,y) \in C_x^1(D)$ with respect to the variable y satisfy HOlder's conditions, $b_{ss}(x,y)$ with respect to the variable x satisfy Holder's conditions, at $j \neq s, a_{js}(x,y) \in C_x^1(D)$ be continuous on y and $b_{js}(x,y) \in C_y^1(D)$ be continuous on x. Then the system (1) can be written in the form:

$$\begin{array}{l} \left[\frac{\partial}{\partial x} + \frac{b_{ss}(x,y)}{x}\right]\left[\frac{\partial}{\partial y} + \frac{a_{ss}(x,y)}{y}\right] U_s(x,y) = \frac{f_s(x,y)}{xy} + \frac{c_s^{(1)}(x,y)}{xy} U_s(x,y) - \\ - \sum\limits_{j=1, j\neq s}^{n} \left[\frac{a_{js}(x,y)}{y} . \frac{\partial U_j(x,y)}{\partial x} + \frac{b_{js}(x,y)}{x} . \frac{\partial U_j(x,y)}{\partial y} + \frac{c_j(x,y)}{x,y} U_j(x,y) \equiv F_s(x,y)\right], \end{array} \tag{11}$$

$$c_s^{(1)}(x,y) = -c_s(x,y) + x\frac{\partial a_{ss}(x,y)}{\partial x} + a_{ss}(x,y).b_{ss}(x,y) \tag{12}$$

Then we can put

$$\frac{\partial U_s(x,y)}{\partial y} + \frac{a_{ss}(x,y)}{y} U_s(x,y) = V_s(x,y). \tag{13}$$

Substituting in equation (11), we get

$$\frac{\partial V_s(x,y)}{\partial x} + \frac{b_{ss}(x,y)}{x} V_s(x,y) = F_s(x,y). \tag{14}$$

Solving equation (14), we get

$$\begin{array}{rl} V_s(x,y) = & e^{-w_1^s(x,y)} |x|^{-b_{ss}(x,0)}.\{\Psi_s(y) + \\ & + \int_x^\infty e^{w_1^s(t,y)}.|t|^{b_{ss}(0,y)}.F_s(t,y)dt\}, \end{array}$$

where $\Psi_s(y)$ are arbitrary continuous functions on Γ_2,

$$w_1^s(x,y) = \int_x^\infty \frac{b_{ss}(t,y) - b_{ss}(0,y)}{t} dt. \tag{15}$$

Similarly, by solving equation (13), we get

$$\begin{array}{rl} U_s(x,y) = & e^{-w_2^s(x,y)} |y|^{-a_{ss}(x,0)}.\{\Phi_s(y) + \\ & + \int_y^\infty e^{w_2^s(x,\tau)}.|\tau|^{a_{ss}(x,0)}.V_s(t,y)d\tau\}, \end{array} \tag{16}$$

where $\Phi_s(x)$ are arbitrary continuous functions,

$$w_2^s(x,y) = \int_y^\infty \frac{a_{ss}(x,\tau) - a_{ss}(x,0)}{\tau} d\tau.$$

Substituting the obtained value $V_s(x,y)$ in equation (16) we get

$$\begin{array}{rl} U_s(x,y) = & e^{-w_2^s(x,y)} |y|^{-a_{ss}(x,0)}.[\Phi_s(x) + \\ & + \int_y^\infty e^{w_2^s(x,\tau)}.|\tau|^{a_{ss}(x,0)}.\{e^{-w_1^s(x,\tau)} |x|^{-b_{ss}(0,\tau)}.(\Psi_s(\tau) + \\ & + \int_x^\infty e^{w_1^s(t,\tau)}.|t|^{b_{ss}(0,\tau)}.F_s(t,\tau)dt\}d\tau]. \end{array} \tag{17}$$

Then we get

$$\begin{aligned}&U_s(x,y)-e^{-w_2^s(x,y)}|y|^{-a_{ss}(x,0)}\int_y^\infty e^{w_2^s(x,\tau)-w_1^s(x,\tau)}.|\tau|^{a_{ss}(x,0)}.d\tau\\&\int_x^\infty e^{w_1^s(t,\tau)}|\tfrac{t}{x}|^{b_{ss}(0,\tau)}.\{\tfrac{c_s^1(t,\tau)}{t\tau}U_s(t,\tau)-\textstyle\sum_{j=1,j\neq s}^n(\tfrac{a_{js}(t,\tau)}{\tau}.\tfrac{\partial U_j(t,\tau)}{\partial t}+\\&\tfrac{b_{js}(t,\tau)}{t}\tfrac{\partial U_j(t,\tau)}{\partial\tau}+\tfrac{c_j(t,\tau}{t\tau}U_j(t,\tau))\}dt=F_s^{(1)}(x,y),\end{aligned}\tag{18}$$

where

$$\begin{aligned}&F_s^{(1)}(x,y)=e^{-w_2^s(x,y)}|y|^{a_{ss}(x,0)}.\{\Phi_s(x)+\\&+\int_y^\infty e^{w_2^s(x,\tau)-w_1^s(x,\tau)}.|\tau|^{a_{ss}(x,0)}|x|^{-b_{ss}(x,0)}.\Psi_s(\tau)d\tau+\int_y^\infty e^{w_2^s(x,\tau)}.\\&.e^{-w_1^s(x,\tau)}|\tau|^{a_{ss}(x,0)}.|x|^{-b_{ss}(0,\tau)}d\tau\int_x^\infty e^{w_1^s(t,\tau)}.|\tau|^{b_{ss}(0,y)}.\tfrac{f_s(t,\tau)}{t\tau}dt\}\end{aligned}\tag{19}$$

and the function $F_s^{(1)}(x,y)$ satisfy: $b_{ss}(0,y)>0, a_{ss}(x,0)>0$.
Also we get

$$\begin{aligned}&\int_y^\infty e^{w_2^s(x,\tau)-w_1^s(x,\tau)}.|\tau|^{a_{ss}(x,0)}d\tau\int_x^\infty e^{w_1^s(t,\tau)}.|\tau|^{b_{ss}(0,\tau)}.|x|^{-b_{ss}(0,\tau)}.\\&\tfrac{a_{js}(t,\tau)}{\tau}\tfrac{\partial U_j(t,\tau)}{\partial t}dt=\int_y^\infty e^{w_2^s(x,\tau)}\tfrac{1}{\tau}.a_{js}(x,\tau)|\tau|^{a_{ss}(x,0)}U_j(x,\tau)d\tau-\\&-\int_y^\infty e^{w_2^s(x,\tau)-w_1^s(x,\tau)}.|\tau|^{a_{ss}(x,0)}.\tfrac{1}{\tau}|x|^{-b_{ss}(0,\tau)}.\\&.\int_x^\infty[\tfrac{\partial a_{js}(t,\tau)}{\partial t}+\tfrac{a_{js}(t,\tau)}{t}.b_{ss}(t,\tau)]e^{w_1^s(t,\tau)}.|t|^{b_{ss}(x,0)}U_j(x,\tau)dt,\end{aligned}\tag{20}$$

$$\begin{aligned}&\int_y^\infty e^{w_2^s(x,\tau)-w_1^s(x,\tau)}.|\tau|^{a_{ss}(x,0)}d\tau\int_x^\infty e^{w_1^s(t,\tau)}.|t|^{b_{ss}(0,\tau)}.|x|^{-b_{ss}(0,\tau)}.\\&\tfrac{b_{js}(t,\tau)}{t}\tfrac{\partial U_j(t,\tau)}{\partial t}dt=e^{w_2^s(x,y)-w_1^s(x,y)}.|y|^{a_{ss}(x,0)}.\\&\int_x^\infty e^{w_1^s(t,\tau)}.|\tfrac{t}{x}|^{b_{ss}(x,0)}.\tfrac{b_{js}(t,y)}{t}.U_j(x,y)dt-\int_y^\infty(\tfrac{a_{ss}(x,\tau)}{\tau}-\tfrac{\partial w_1^s(x,\tau)}{\partial\tau})d\tau\\&\int_x^\infty e^{w_1^s(t,\tau)}.|t|^{b_{ss}(0,\tau)}.|x|^{-b_{ss}(0,\tau)}.\tfrac{b_{js}(t,\tau)}{t}U_j(t,\tau)-\\&-\int_y^\infty e^{w_2^s(x,\tau)-w_1^s(x,\tau)}.|\tau|^{a_{ss}(x,0)}d\tau.\\&.\int_x^\infty[\tfrac{\partial w_1^s(t,\tau)}{\partial\tau}.b_{js}(t,\tau)+\tfrac{\partial b_{ss}(0,\tau)}{\partial\tau}.b_{js}(t,\tau)(ln|\tfrac{t}{x}|)+\tfrac{\partial b_{js}(t,\tau)}{\partial\tau}]t^{-1}.\\&.|\tfrac{t}{x}^{b_{ss}(0,\tau)}.e^{w_1^s(t,\tau)}.U_j(x,\tau)dt.\end{aligned}\tag{21}$$

Substituting the obtained values in equation (18) we get

$$\begin{aligned}&K_1^{(1)}(x,y;t,\tau)=e^{-w_2^s(x,y)}.|y|^{a_{ss}(x,0)}.\Big\{e^{w_2^s(x,\tau)}.\\&.e^{-w_1^s(x,\tau)+w_1^s(x,0)}.|\tau|^{a_{ss}(x,0)}.|\tfrac{t}{x}|^{b_{ss}(0,\tau)}(t\tau)^{-1}c_s^{(1)}(t,\tau),\end{aligned}$$

$$\begin{aligned}&K_{js}^{(2)}(x,y;t,\tau)=e^{-w_2^s(x,y)}.|y|^{a_{ss}(x,0)}.\\&\Big\{-[e^{w_2^s(x,\tau)-w_1^s(x,\tau)}.|\tau|^{a_{ss}(x,0)}\tfrac{1}{\tau|x|^{b_{ss}(0,\tau)}}.\\&(\tfrac{\partial a_{js}(t,\tau)}{\partial t}+\tfrac{a_{js}(t,\tau)}{t}b_{ss}(t,\tau)).\\&.e^{w_1^s(t,\tau)}.|t|^{b_{ss}(0,\tau)}-(\tfrac{a_{js}(x,\tau)}{\tau}-\tfrac{\partial w_1^s(x,\tau)}{\partial\tau}).\\&.e^{w_1^s(t,\tau)}.|t|^{b_{ss}(0,\tau)}|x|^{b_{ss}(0,\tau)}.(\tfrac{b_{js}(t,\tau)}{\partial\tau}-\\&-e^{w_2^s(x,\tau)-w_1^s(x,\tau)+w_1^s(t,\tau)}>|\tau|^{a_{ss}(x,0)}.\\&.(\tfrac{\partial w_1^s(t,\tau)}{\partial\tau}b_{js}(t,\tau)+\tfrac{\partial b_{ss}(0,\tau)}{\partial\tau}b_{js}(t,\tau)ln|\tfrac{t}{x}|+\tfrac{\partial b_{js}(t,\tau)}{\partial\tau})\\&t^{-1}.|t|^{b_{ss}(0,\tau)}|x|^{-b_{ss}(0,\tau)}]\Big\},\end{aligned}\tag{22}$$

$$K_{js}^{(3)}(x,y;t) = e^{w_1^s(t,y)-w_1^s(x,y)}.|\tfrac{t}{x}|^{b_{ss}(0,y)}.\tfrac{b_{js}(t,y)}{t},$$
$$K_{js}^{(4)}(x,y;\tau) = e^{w_2^s(x,\tau)-w_2^s(x,y)}.|y|^{-a_{ss}(x,0)}.\tfrac{|\tau|^{a_{ss}(x,0)}}{\tau}a_{js}(x,\tau),$$

then we get the solution of the following system integral equations:

$$U_s(x,y) - \int_y^\infty d\tau \int_x^\infty K_1^{(1)}(x,y;t,\tau)U_s(t,\tau)dt-$$

$$\sum_{j=1,j\neq s}\{\int_y^\infty d\tau \int_x^\infty K_{js}^{(2)}(x,y;t,\tau)U_j(t,\tau)dt+$$

$$+\int_x^\infty K_{js}^{(3)}(x,y;t)U_j(t,y)dt + \int_y^\infty K_{js}^{(4)}(x,y;\tau)U_j(t,\tau)d\tau\} = F_s^{(1)}(x,y), 1\leq s\leq n. \tag{23}$$

Then Kernels satisfy the following properties:

$$K_1^{(1)}(x,y;x,\tau) = e^{w_2^s(x,\tau)-w_2^s(x,y)}.|\tfrac{\tau}{y}|^{a_{ss}(x,0)}(x\tau)^{-1}c_s^{(1)}(t,\tau),$$
$$K_1^{(1)}(x,y;x,y) = \tfrac{c_s^{(1)}(x,y)}{xy},$$

$$\begin{aligned}K_{js}^{(2)}(x,y;x,\tau) &= e^{w_2^s(x,\tau)-w_2^s(x,y)}[|y|^{-a_{ss}(x,0)}.\\ &|\tau|^{a_{ss}(x,0)}\tau^{-1}(\tfrac{\partial a_{js}(x,\tau)}{\partial x}+\tfrac{a_{js}(x,\tau)}{x}b_{ss}(x,\tau))-\\ &-(\tfrac{a_{ss}(x,\tau)}{\tau}-\tfrac{\partial w_1^s(x,\tau)}{\partial\tau}).\tfrac{b_{js}(x,\tau)}{x}-|\tau|^{a_{ss}(x,0)}.|y|^{-a_{ss}(x,0)}.\\ &.(\tfrac{\partial w_1^s(x,\tau)}{\partial\tau}b_{js}(x,\tau)+\tfrac{\partial b_{js}(x,\tau)}{\partial\tau})x^{-1}],\end{aligned}$$

$$\begin{aligned}K_{js}^{(2)}(x,y;x,y) &= \tfrac{1}{y}\tfrac{\partial a_{js}(x,y)}{\partial x}+\tfrac{a_{js}(x,y)}{xy}b_{ss}(x,y)-\tfrac{a_{ss}(x,y)}{xy}b_{js}(x,y)+\\ &+\tfrac{1}{x}\tfrac{\partial w_1^s(x,y)}{\partial y}-\tfrac{1}{x}b_{js}(x,y)\tfrac{\partial w_1^s(x,y)}{\partial y}-\tfrac{1}{x}\tfrac{\partial b_{js}(x,y)}{\partial y}=\\ &=\frac{x\frac{\partial a_{js}(x,y)}{\partial x}-y\frac{\partial b_{js}(x,y)}{\partial y}}{xy}+\frac{y\frac{\partial w_1^s(x,y)}{\partial y}-yb_{js}(x,y)\frac{\partial w_1^s(x,y)}{\partial y}}{xy}, j\neq s,\end{aligned}$$

$$K_{js}^{(3)}(x,y;x) = \frac{b_{js}(x,y)}{x}, \tag{24}$$

$$K_{js}^{(4)}(x,y;x) = \frac{a_{js}(x,y)}{y}, j\neq s. \tag{25}$$

Equation (19) satisfy

$$0 < a_{ss}(x,0) < 1,\ 0 < b_{ss}(0,y) < 1.$$

We introduce the new unknown function:

$$V_s(x,y) = U_s(x,y)|y|^{a_{ss}(x,0)}e^{w_2^s(x,y)}.$$

For $V_s(x,y)$, we can get the general solution for equation (1) in the form of integral equation (9), where

$$M_1^{(1)}(x,y;t,\tau)=K_1^{(1)}(x,y;t,\tau)e^{-w_2^s(x,\tau)}.|\tau|^{-a_{ss}(t,0)}=$$
$$=e^{w_2^s(x,\tau)-w_2^s(t,\tau)+w_1^s(x,0)-w_1^s(x,\tau)}.|\tau|^{a_{ss}(x,0)-a_{ss}(t,0)}.$$
$$.|\tfrac{t}{x}|^{b_{ss}(0,\tau)}(t\tau)^{-1}c_s^{(1)}(t,\tau),$$

$$M_{js}^{(2)}(x,y;t,\tau)=K_{js}^{(2)}(x,y;t,\tau)e^{-w_2^j(t,\tau)}.|\tau|^{-a_{jj}(t,0)}=$$
$$=-\{e^{w_2^s(x,\tau)-w_2^j(t,\tau)+w_1^s(t,\tau)-w_1^s(x,\tau)}.|\tau|^{a_{ss}(x,0)-a_{jj}(t,0)}.$$
$$.|\tfrac{t}{x}|^{b_{ss}(0,\tau)}[\frac{t\frac{\partial a_{js}(t,\tau)}{\partial t}+a_{js}(t,\tau)}{t\tau}]-e^{w_1^s(t,\tau)-w_2^j(t,\tau)}.|\tfrac{t}{x}|^{b_{ss}(0,\tau)}$$
$$|\tau|^{-a_{jj}(t,0)}\frac{b_{js}(t,\tau)}{t}[\frac{a_{ss}(x,\tau)}{\tau}-\frac{\partial w_1^s(x,\tau)}{\partial\tau}]-$$
$$-e^{w_2^s(x,\tau)-w_2^j(t,\tau)+w_1^s(t,\tau)-w_1^s(x,\tau)}.|\tau|^{a_{ss}(x,0)-a_{jj}(t,0)}.$$
$$.|\tfrac{t}{x}|^{b_{ss}(x,0)}.\frac{[\frac{\partial w_1^s(t,\tau)}{\partial\tau}b_{js}(t,\tau)+\frac{\partial b_{ss}(0,\tau)}{\partial\tau}b_{js}(t,\tau)ln|\frac{t}{x}|+\frac{\partial b_{js}(t,\tau)}{\partial\tau}]}{t}\},$$

$$M_{js}^{(3)}(x,y;t)=K_{js}^{(3)}(x,y;t)e^{-w_2^j(t,\tau)}.|\tau|^{-a_{jj}(t,0)}=$$
$$=e^{w_2^s(x,y)-w_2^j(t,\tau)+w_1^s(t,y)-w_1^s(x,y)}.|\tfrac{t}{x}|^{b_{ss}(0,y)}$$
$$.|y|^{a_{ss}(x,0)-a_{jj}(t,0)}.b_{js}(t,y)t^{-1},$$

$$M_{js}^{(4)}(x,y;\tau)=K_{js}^{(4)}(x,y;\tau)=e^{-w_2^j(x,\tau)}.|\tau|^{-a_{jj}(x,0)}=.$$
$$=e^{w_2^s(x,\tau)-w_2^j(x,\tau)}.|\tau|^{a_{ss}(x,0)-a_{jj}(t,0)}.a_{js}(t,y).\tau^{-1}.$$

The Kernels satisfy the following properties:

$$M_1^{(1)}(x,y;x,y)=\frac{c_s^{(1)}(x,y)}{xy},$$
$$M_{js}^{(2)}(x,y;x,y)=e^{w_2^s(x,y)-w_2^j(x,y)}.|y|^{a_{ss}(x,0)-a_{jj}(x,0)}$$
$$[\frac{(x\frac{\partial a_{js}(x,y)}{\partial x}+a_{js}(x,y))}{xy}-\frac{(a_{ss}(x,y)-y\frac{\partial w_1^s(x,y)}{\partial y})}{xy}].$$
$$.e^{w_1^s(x,y)-w_2^j(x,y)}.b_{js}(x,y)|y|^{a_{jj}(x,0)}-$$
$$-[\frac{\frac{\partial w_1^s(x,y)}{\partial y}b_{js}(x,y)+\frac{\partial b_{js}(x,y)}{\partial y}}{x}].e^{w_2^s(x,y)-w_2^j(x,y)},$$

$$M_{js}^{(3)}(x,y;x)=e^{w_2^j(x,y)}.|y|^{a_{ss}(x,0)}e^{w_2^s(x,y)}\frac{b_{js}(x,y)}{x},$$
$$M_{js}^{(3)}(x,y;y)=\frac{a_{js}(x,y)}{y}.|y|^{a_{ss}(x,0)-a_{jj}(x,0)}.e^{w_2^s(x,y)-w_2^j(x,y)}.$$

From the above inequalities, if the coefficients of the system (1) satisfy the conditions of equation (9) on $V_s(x,y)$ and satisfy the conditions:
1) $c_1^s(x,y)$ in a neighborhood Γ_1,Γ_2 satisfy condition (2),

2) $$\beta_{js}+\beta_{js}^{(1)}>a_{jj}(x,0),\alpha_{js}^{(1)}>a_{jj}(x,0)-a_{ss}(x,0) \tag{26}$$

3) The coefficients $b_{js}(x,y)$ are continuous with the variable y,
4) $\frac{\partial b_{ss}(x,y)}{\partial y}$ satisfy Holder's conditions with the variable x.

Then the system (1) is the Volterra integral equations of the second type. By solving system (9), we get $V_s(x,y)$ and substituting the obtained value $V_s(x,y)$ in equation (8), we get the solution $U_s(x,y)$ of the system (1).
The proof is completed.

Theorem 2 *Let the coefficients in system (1) satisfy the following conditions:*

a) $a_{js}(x,y)(j \neq s)$ *are continuous with the variable* y *and have continuous derivatives of the first order with the variable* $x, b_{js}(x,y)(j \neq s)$ *are continuous with the variable* x *and have continuous derivatives of the first order with the variable* y.

b) $a_{js}(x,y)(j \neq s)$ *are continuous with the variable* y *and have continuous derivatives of the first order with the variable* x.

c) $b_{ss}(x,y) \in C(\bar{D}), f(x,y) \in C(\bar{D}), c_s(x,y) \in C(\bar{D})$.

d) In the neighborhood Γ_1 *and* Γ_2 *the functions satisfy the conditions (7), (23).*

e) $0 < a_{ss}(x,0) < 1,\ 0 < b_{ss}(0,y) < 1,\ 1 \leq s \leq n.$

Then any solution of the system (1) within the class $C^2(D)$ *is:*

$$U_s(x,y) = e^{-w_1^s(x,y)}|x|^{-b_{ss}(0,y)}.V_s(x,y), \tag{27}$$

where $V_s(x,y)$ *is the solution of system (1) Volterra integral equations of the second type in the following form:*

$$V_s(x,y) + \int_x^\infty dt \int_y^\infty K_s^{(1)}(x,y;t,\tau)V_s(t,\tau)d\tau +$$
$$\int_{j=1, j\neq s}^{n} \{ \int_x^\infty dt \int_y^\infty K_{js}^{(2)}(x,y;t,\tau)V_j(t,\tau)d\tau +$$
$$+ \int_{js}^{(3)}(t,y)V_j(t,y)dt + \int_y^\infty K_{js}^{(4)}(x,y;\tau)V_j(t,\tau)d\tau\} = F_s^{(2)}(x,y), 1 \leq s \leq n, \tag{28}$$

where $\Phi_s^{(1)}(x), \Psi_s^{(1)}(y)$ *are arbitrary continuous functions of the variables* x *and* y. *Moreover;* $\Phi_s^{(1)}(x) \in C^1(\Gamma_1), \Psi_s^{(1)}(y) \in C^2(\Gamma_2)$,

$$\begin{array}{l} F_s^{(2)}(x,y) = \Psi_s^{(1)}(y) + \int_x^\infty e^{w_1^s(t,y) - w_2^j(t,y)}|y|^{a_{ss}(t,0)}.|t|^{b_{ss}(0,y)} \\ .\Phi_s^{(1)}(t)dt + \int_s^\infty e^{w_1^s(t,y) - w_2^s(t,y)}.|y|^{-a_{ss}(t,0)}.|t|^{b_{ss}(0,y)}dt. \\ .\int_y^\infty e^{w_2^s(t,\tau)}.|\tau|^{a_{ss}(t,0)}.\frac{f_s(t,\tau)}{t\tau}dt. \end{array} \tag{29}$$

Proof. Let the coefficients in system (1) satisfy: $b_{ss}(x,y) \in C^1_y(D)$ with respect to the variable x satisfy Holder's conditions, $a_{ss}(x,y)$ with respect to the variable y satisfy Holder's conditions, at $j \neq s$ $a_{js}(x,y) \in C^1_x(D)$ be continuous with respect to the variable y and $b_{js}(x,y) \in C^1_y(D)$ be continuous with respect to the variable x. Then the system (1) can be written in the form:

$$\begin{gathered}\left[\frac{\partial}{\partial y} + \frac{a_{ss}(x,y)}{y}\right]\left[\frac{\partial}{\partial x} + \frac{b_{ss}(x,y)}{x}\right] U_s(x,y) = \frac{f_s(x,y)}{xy} + \frac{c_s^{(2)}(x,y)}{xy}.U_s(x,y)- \\ -\sum_{j=1}^{n}\left[\frac{a_{js}(x,y)}{y}\frac{\partial U_j(x,y)}{\partial x} + \frac{b_{js}(x,y)}{x}\frac{\partial U_j(x,y)}{\partial y} + \frac{c_j(x,y)}{xy}U_j(x,y) \equiv F_s^{(1)}(x,y)\right],\end{gathered} \tag{30}$$

where

$$c_x^{(2)}(x,y) = -c_s(x,y) + y\frac{\partial b_{ss}(x,y)}{\partial y} + a_{ss}(x,y).b_{ss}(x,y). \tag{31}$$

By solving equation (30), we get

$$\begin{gathered}U_s(x,y) - e^{-w_1^s(x,y)}|x|^{-b_{ss}(0,y)}.e^{w_1^s(t,y)-w_2^s(t,y)}.|t|^{b_{ss}(0,y)}dt. \\ .\int_y^\infty e^{w_2^s(x,\tau)}|\tfrac{\tau}{y}|^{a_{ss}(t,0)}.\{\tfrac{c_s^{(2)}(t,\tau)}{t\tau}U_s(t,\tau) - \sum_{j=1,j\neq s}^{n}(\tfrac{a_{js}(t,\tau)}{\tau}.\tfrac{\partial U_j(t,\tau)}{\partial t}+ \\ +\tfrac{b_{js}(t,\tau)}{t}.\tfrac{\partial U_j(t,\tau)}{\partial \tau} + \tfrac{c_j(t,\tau)}{t\tau}U_j(t,\tau))\}dt = F_s^{(1)}(x,y),\end{gathered} \tag{32}$$

$$\begin{gathered}F_s^{(1)(x,y)} = e^{-w_1^s(x,y)}|x|^{-b_{ss}(0,y)}.\{\Psi_s^{(1)}(y)+ \\ +\int_x^\infty e^{w_1^s(t,y)-w_2^s(t,y)}.|y|^{-a_{ss}(t,0)}.|t|^{b_{ss}(0,y)}dt.\int_y^\infty e^{w_2^s(t,\tau)}.|\tau|^{a_{ss}(t,0)}. \\ \tfrac{f_s(t,\tau)}{t\tau}d\tau\}.\end{gathered} \tag{33}$$

Similarly, we can get

$$\begin{gathered}U_s(x,y).e^{w_1^s(x,y)}|x|^{b_{ss}(0,y)} = V_s(x,y), \\ K_s^{(1)}(x,y;t,\tau) = -e^{w_2^s(t,\tau)-w_2^s(t,y)+w_1^s(t,y)-w_1^s(t,\tau)}. \\ .|t|^{b_{ss}(0,y)-b_{ss}(0,\tau)}.|\tfrac{\tau}{y}|^{a_{ss}(t,0)}(x\tau)^{-1}c_2^s(t,\tau),\end{gathered}$$

$$\begin{gathered}K_{js}^{(2)}(x,y;t,\tau) = e^{w_1^s(t,y)-w_1^j(t,\tau)+w_2^s(t,\tau)-w_2^s(t,y)}. \\ .|t|^{b_{ss}(0,y)-b_{jj}(0,\tau)}.[\tfrac{a_{js}(t,\tau)}{\tau}.(\tfrac{\partial w_2^s(t,\tau)}{\partial t} + \tfrac{\partial a_{ss}(t,0)}{\partial t}.ln|\tfrac{\tau}{y}|) + \tfrac{1}{\tau}\tfrac{\partial a_{js}(t,\tau)}{\partial \tau}]. \\ -e^{w_1^s(t,y)-w_2^s(t,y)+w_2^s(t,\tau)-w_1^j(t,\tau)}.(-\tfrac{\partial w_2^s(t,y)}{\partial t} + \tfrac{b_{ss}(t,y)}{t}). \\ .|t|^{b_{ss}(0,y)-b_{jj}(0,\tau)}.|\tfrac{\tau}{y}|^{a_{ss}(t,0)}.\tfrac{a_{js}(t,\tau)}{\tau}- \\ -e^{w_1^s(t,y)-w_2^s(t,y)+w_2^s(tr,\tau)-w_1^j(t,\tau)}.|t|^{b_{ss}(0,y)-b_{jj}(0,\tau)}.[(\tfrac{a_{ss}(t,0)}{\tau}) \\ .\tfrac{b_{js}(t,\tau)}{t} + \tfrac{1}{t}\tfrac{\partial b_{js}(t,y)}{\partial \tau}].|\tfrac{\tau}{y}|^{a_{ss}(t,0)} + \tfrac{c_j(t,\tau)}{t\tau}.|t|^{-b_{jj}(0,\tau)}.e^{-w_1^j(t,\tau)},\end{gathered}$$

$$K_{js}^{(3)}(t,y) = e^{w_1^s(t,y)-w_1^j(t,y)}.\frac{1}{t}[|t|^{b_{ss}(0,y)-b_{jj}(0,y)}.b_{js}(t,y)],$$

$$K_{js}^{(4)}(t,y,\tau) = \quad e^{w_1^s(x,y)-w_2^s(x,y)+w_2^s(x,\tau)-w_1^j(x,\tau)}.|x|^{b_{ss}(0,y)-b_{jj}(0,\tau)}.$$
$$.|\tfrac{\tau}{y}|^{a_{ss}(x,0)}.\tfrac{a_{js}(x,\tau)}{\tau}.$$

Moreover the Kernels of the system (26) at $x = t, y = \tau$ satisfy the following properties:

$$K_s^{(1)}(x,y;x,y) = \frac{c_2^s(x,y)}{y},$$

$$K_{js}^{(2)}(x,y;x,y) = e^{w_1^s(x,y)-w_1^j(x,y)}.[\tfrac{a_{js}(x,y)}{y}.\tfrac{\partial w_2^s(x,y)}{\partial x} + \tfrac{1}{y}\tfrac{\partial a_{js}(x,y)}{\partial x}]$$
$$|x|^{b_{ss}(0,y)-b_{jj}(0,y)} - (-\tfrac{\partial w_2^s(x,y)}{\partial x} - \tfrac{b_{ss}(x,y)}{x})|x|^{b_{ss}(0,y)-b_{jj}(0,y)}.$$
$$\tfrac{a_{js}(x,y)}{y} - [\tfrac{a_{ss}(x,0)}{y}.\tfrac{b_{js}(x,y)}{x} + \tfrac{1}{x}\tfrac{\partial b_{js}(x,y)}{\partial y}] + \tfrac{c_j(x,y)}{xy}|x|^{-b_{jj}(0,y)}.$$
$$.e^{-w_1^j(x,y)}.$$

The function $K_{js}^{(2)}(x,y;x,y)$ can be represented in the form

$$K_{js}^{(2)}(x,y;x,y) = [\frac{A_{js}^{(2)}(x,y)}{y}. + \frac{B_{js}^{(2)}(x,y)}{xy}]|x|^{b_{ss}(0,y)-b_{js}(0,y)},$$

where

$$A_{js}^{(2)}(x,y) = a_{js}(x,y)\frac{\partial w_2^s(x,y)}{\partial x} + \frac{\partial a_{js}(x,y)}{\partial x},$$

$$B_{js}^{(2)}(x,y) = (x\tfrac{\partial w_2^s(x,y)}{\partial x} + b_{ss}(x,y)).a_{js}(x,y) - a_{ss}(x,0).b_{js}(x,y) + y\tfrac{\partial b_{js}(x,y)}{\partial y}+$$
$$+|x|^{b_{ss}(0,y)-b_{jj}(0,y)}c_j(x,y).e^{-w_1^j(x,y)},$$

$$K_{js}^{(3)}(x,y) = \frac{b_{js}(x,y)}{x}.|x|^{b_{ss}(0,y)-b_{jj}(0,y)}.e^{w_1^s(t,y)-w_1^j(t,y)},$$

$$K_{js}^{(4)}(x,y;y) = \frac{a_{js}(x,y)}{y}.|x|^{b_{ss}(0,y)-b_{jj}(0,y)}.e^{w_1^s(x,y)-w_1^j(x,y)}.$$

The system (28) is the Volterra system integral equation of the second type if the functions $a_{js}(x,y), b_{js}(x,y), c_2^s(x,y)$ satisfy the following conditions:

$$c_2^s(x,y) = 0(|x|^{\alpha_s}),\ \alpha_s > 0 \quad \text{in a neighborhood of } \Gamma_1, \tag{34}$$

$$c_2^s(x,y) = 0(|x|^{\beta_s}),\ \beta_s > 0 \quad \text{in a neighborhood of } \Gamma_2, \tag{35}$$

$$|b_{ss}(x,y) - b_{ss}(0,y)| \le H_1|x|^{\delta_s},\ \delta_s > 0 \tag{36}$$

$$a_{js}(x,y) = 0(|y|^{\beta_{js}^{(2)}}),\ j \neq s, \beta_{js}^{(2)} > 0 \text{ in a neighborhood of } \Gamma_2, \tag{37}$$

$$A_{js}^{(2)}(x,y) = 0(|y|^{\beta_{js}^{(2)}}), \beta_{js}^{(2)} > 0, j \neq s \text{ in a neighborhood of } \Gamma_2, \tag{38}$$

$$B_{js}^{(2)}(x,y) = 0(|y|^{\delta_{js}^{(2)}}), \delta_{js} > 0, j \neq s \text{ in a neighborhood of } \Gamma_2, \tag{39}$$

$$B_{js}(x,y) = 0(|x|^{\delta_{js}^{(3)}}) > 0, j \neq s \quad \text{in a neighborhood of } \Gamma_1, \tag{40}$$

Also the functions $a_{ss}(x,y), b_{ss}(x,y)$ satisfy the following conditions:

$$|a_{ss}(x,y) - a_{ss}(x,0)| \leq H_2|y|^{\delta_s}, \delta_s > 0 \tag{41}$$

$$|\frac{\partial a_{ss}(x,y)}{\partial x} - \frac{\partial a_{ss}(x,0)}{\partial x}| \leq H_2|y|^{\beta_s^{(1)}}, \beta_s^{(1)} > 0, \tag{42}$$

By solving the system (28), we get $F_s^{(2)}(x,y)$. Substituting the obtained value $F_s^{(2)}(x,y)$ in equation (28), we get the solution $U_s(x,y)$ of the system (1).
The proof is completed.

Remak 1 *The coefficients of system (1) in a neighborhood of* Γ_1, Γ_2 *satisfy the conditions (27)-(32).*
Example *Let in Theorem 1 and Theorem 2 the conditions:*

$$c_s^{(1)}(x,y) = 0(|x|^{\alpha_s^1}), \alpha_s^2 > 0, c_s^{(1)}(x,y) = 0(|y|^{\beta_s^1}), \beta_s^1 > 0$$

in a neighborhood of Γ_1 *are essential and the satisfy the conditions*

$$c_s^{(2)}(x,y) = 0(|x|^{\alpha_s^2}), \alpha_s^2 > 0, c_s^{(2)}(x,y) = 0(|y|^{\beta_s^2}), \beta_s^2 > 0.$$

If these conditions are fulfilled, then system (1) have solution essentially different from the solution given in Theorem 1 and Theorem 2.
Let in system (1) $a_{ss} = \alpha =$ *constant,* $b_{ss} = \beta =$ *constant,* $f_s(x,y) = 0$*, then the functions of the form*

$$\begin{aligned} U(x,y) = \quad & x^{-\beta} \textstyle\sum_{n=0}^{\infty} y^{\lambda n}.x^{\frac{\alpha\beta-\gamma}{\lambda n+\alpha}}.\beta_{\lambda n} + y^{-\alpha}. \textstyle\sum_{n=0}^{\infty} x^{\lambda n}. \\ & .\lambda^{-\frac{\alpha\beta-\gamma}{\lambda n+\beta}}.A_{\lambda n}, \alpha, \beta \neq -\lambda_n, \end{aligned} \tag{43}$$

are the solutions of equations (1), where $n = \lambda_n, A_{\lambda n}, \beta_{\lambda n}$ *being arbitrary constants and the series converges at* $0 < x < A^{-1}, 0 < y < \beta^{-1}$,

$$A = \lim_{n\to\infty} |\frac{A_{\lambda n+1}}{A_{\lambda n}}|, \qquad B = \lim_{n\to\infty} |\frac{B_{\lambda n+1}}{B_{\lambda n}}|.$$

In particular if $\alpha\beta = \gamma$*, then the general solution of equation (1) is given by the formula:*

$$U(x,y) = x^{-\beta}\Phi(y) + y^{-\alpha}\Psi(x),$$

where $\Phi(y), \Psi(x)$ *are arbitrary continuous different functions.*
Problem A_1**.** *Find a solution of system (1) within the class* $C^2(D)$ *and the boundary conditions:*

$$\begin{aligned} U_s(x,0) &= f_s^1(x), x \in \Gamma_1, \\ U_s(x,0) &= g_s^1(x), y \in \Gamma_2, \end{aligned} \tag{44}$$

where $f_s^1(x), g_s^1(y)$ *are arbitrary continuous functions.*
Solution of Problem A_1 *In the case where* $C_s^{(1)}(x,y) = 0$*. Using the integral representations (17), (18) and the conditions of equation (40) we can get the values* $Phi_s(x), \Psi_s(y), 1 \le s \le n$ *in the form*

$$g_s^1(y)|y|^{b_{js}(y,y)} = \Psi_s(y),$$

$$\Phi_s(x) = |x|^{-a_{js}(x,x)} \left[f_s^1(x) + \frac{1}{x} b_{js}(x,0).f_s(x) \right], \tag{45}$$

and by (45), the condition f) of Theorem 1 where $\Phi_s(x) \in C(\Gamma_1)$*,* $\Psi_s(y) \in C^1(\Gamma_2)$*, we get* $f_s^1(x) \in C(\Gamma_1), g_s^1(y) \in C^1(\Gamma_2)$*.*
Solving the system of integral equations (9), we get $V_s(x,y)$ *and substituting the obtained values* $\Phi_s(x), \Psi_s(y), V_s(x,y)$ *in equation (8) we get the solution of Problem* A_1*.*

Theorem 3 *Let the coefficients of system (1) satisfy the conditions a), b), e), f), equation (12) and in problem* A_1 *the function* $f_s^1(x) \in C(\Gamma_1), g_s^1(y) \in C^1(\Gamma_2)$*. Then problem* A_1 *has the unique solution which is given by the formulae (8), (9), (45).*

Remak 2 *In the case where* $C_s^{(1)}(x,y) \neq 0$*.*
To solve problem A_1*, we use the formula (25) and Remark 1, we can get the unknown functions* $\Phi_s(x), \Psi_s(y)$ *which can be given by the formula (45).*

Theorem 4 *Let the coefficients of system (1) satisfy: The conditions of Theorem 2, Theorem 3 and the functions* $f_s^1(x), g_s^1(y)$ *satisfy the conditions of Theorem 3. Then problem* A_1 *has the unique solution, which is given by the equation (8),* $V_s(x,y)$ *is the solution of the system (9) and the functions* $\Phi_s(x), \Psi_s(y)$ *are given by the formula (45).*

Problem A_2**.** Find a solution of system (1) within the class $C^2(D)$ and the boundary conditions:

$$U_s(0,y) = g_s^2(y), y \in \Gamma_2, \quad \frac{\partial U_s(x,y)}{\partial x}|_{y=0} = f_s^2(x), x \in \Gamma_1 \tag{46}$$

where $f_s^1(x), g_s^1(y)$ are arbitrary continuous functions.
Solution of Problem A_2. Similarly, as the Solution of Problem A_1, using the integral representations (32), (33) and the conditions of equations (46) we can get the values $\varphi_s^1(x), \psi_s^1(y), 1 \leq s \leq n$ and the values of the condition e) of Theorem 2 and solving the equation (22), we get $U_s(x,y)$ and substituting the obtained values in equation (27) we get the solution of Problem A_2.

Case 2

Also, for system (1), the following statements being valid.

Theorem 5 *Let the coefficients in system (1) satisfy:*
$a_{js}(x,y) \in C_x^1(D), b_{js}(x,y) \in C_y^1(D), a_{ss}(x,y) \in C_x^1(D), b_{ss}(x,y),$
$c_s(x,y) \in C(D), 1 \leq s \leq n,$ *at* $j \neq s, j, s = 1, 2, \ldots, n$*. Then any solution for the system (1) within the class* $C^2(D) \cap C(\overline{D})$ *is:*

$$\begin{gathered}U_s(x,y) - \int_y^\infty e^{w_2^s(x,\tau)-w_2^s(x,y)}d\tau \int_x^\infty c_1^s(t,\tau)U_s(t,\tau).e^{w_1^s(t,\tau)-w_1^s(x,\tau)}dt+ \\ +\int_y^\infty e^{w_2^s(x,\tau)-w_2^s(x,y)}d\tau \int_x^\infty e^{w_1^s(t,\tau)-w_1^s(x,\tau)}. \\ .[-\sum_{\substack{j=1 \\ j\neq s}}^{n}(a_{js}(t,\tau).\frac{\partial U_j}{\partial t} + b_{js}(t,\tau)\frac{\partial U_j}{\partial \tau} + c_j(t,\tau)U_j(t,\tau))]dt = g_s(x,y), 1 \leq s \leq n,\end{gathered} \tag{47}$$

where $\Psi_s(x), \Phi_s(y)$ *are given continuous functions on* Γ_1, Γ_2 *and* $g_s(x,y)$ *is a solution of the system (1) Volterra integral equation of the second type in the form:*

$$\begin{gathered}g_s(x,y) = \Psi_s(x)e^{-w_2^s(x,y)} + \int_y^\infty e^{w_2^s(x,\tau)-w_2^s(x,y)-w_1^s(x,\tau)}.\Phi_s(\tau)d\tau+ \\ +\int_y^\infty e^{w_2^s(x,\tau)-w_2^s(x,y)}d\tau. \int_x^\infty e^{w_1^s(t,\tau)-w_1^s(x,\tau)}f_s(t,\tau)dt,\end{gathered} \tag{48}$$

$$w_1^s(x,y) = \int_x^\infty b_{ss}(t,y)dt, \qquad w_2^s(x,y) = \int_y^\infty a_{ss}(x,\tau)d\tau, \tag{49}$$

Problem A_3. Find a solution of system (1) within the class $C^2(D) \cap C(D \cup \Gamma_1 \cup \Gamma_2)$ with the boundary conditions:

$$\begin{gathered}U_s(0,y) = a_s(y), y \in \Gamma_2, \\ U_s(x,0) = b_s(x), x \in \Gamma_1, 1 \leq s \leq n. \\ a_s(0) = b_s(0).\end{gathered} \tag{50}$$

Solution of Problem A_3. The function $g_s(x,y)$ of equation (48) on Γ_1 satisfy the following properties:

$$U_s(0,y) - \int_y^\infty \left[(M_{js}^{(1)}(0,y;\tau)U_s(0,\tau) - M_{js}^{(2)}(0,y;\tau)U_j(0,\tau))\right] d\tau = g_s(0,y),$$

$$g_s(0,y) = \Psi_s(0).e^{-w_1^s(0,\tau)} + \int_y^\infty e^{w_2^s(0,\tau)-w_2^s(0,y)}.\Phi_s(\tau)d\tau \tag{51}$$

where

$$M_{js}^{(1)}(x,y;\tau) = e^{w_2^s(x,\tau)-w_2^s(x,y)}.a_{js}(x,\tau), \tag{52}$$

$$M_{js}^{(2)}(x,y;\tau) = e^{w_2^s(x,\tau)-w_2^s(x,y)-w_1^s(x,\tau)}.a_{js}(0,\tau), \tag{53}$$

$$M_{js}^{(3)}(x,y;\tau) = e^{w_1^s(t,y)-w_1^s(x,y)}.b_{js}(t,y), \tag{54}$$

$$M_{js}^{(4)}(x,y;t) = e^{w_1^s(t,0)-w_2^s(x,y)-w_1^s(x,0)}.b_{js}(t,0),$$

From equation (52, (53) we get

$$M_{js}^{(1)}(0,y;\tau) = M_{js}^{(2)}(0,y;\tau)$$

then

$$U_s(0,y) = g_s(0,y)$$

Similarly, the function $g_s(x,y)$ of equation (50) on Γ_2 satisfy:

$$U_s(x,0) - \sum_{j=1,j\neq s} \int_x^\infty \left[M_{js}^{(3)}(x,0;t) - M_{js}^{(4)}(x,0;t)\right] U_j(t,0)dt = g_s(x,0), \tag{55}$$

$$g_s(x,0) = \Psi_s(x), 1 \leq s \leq n. \tag{56}$$

Then we can put

$$M_{js}^{(3)}(x,0;t) = e^{w_1^s(t,0)-w_1^s(x,0)}.b_{js}(x,0),$$

$$M_{js}^{(4)}(x,0;t) = e^{w_1^s(t,0)-w_1^s(x,0)}.b_{js}(t,0) = M_{js}^{(3)}(x,0;t).$$

We get

$$U_s(x,0) = g_s(x,0) = \Psi_s(x).$$

Substitute the values $U_s(x,0), U_s(0,y)$ in equations (51), (55) we can get the values $\Psi_s(x), \Phi_s(y)$ in the forms

$$\begin{array}{l} \Psi_s(x) = b_s(x), \\ \int_y^\infty e^{w_2^s(0,\tau)-w_2^s(0,y)}.\Phi_s(tau)d\tau = a_s(y) - b_s(0).e^{-w_2^s(0,y)}, \\ \int_y^\infty e^{w_2^s(0,\tau)}.\Phi_s(\tau)d\tau = a_s(y).e^{w_2^s(0,y)} - b_s(0). \end{array}$$

$$\Phi_s(y) = e^{-w_2^s(0,y)}\left[e^{w_2^s(0,y)}.a_{ss}(0,y)(a_s(y) - b_s(0)) + e^{w_2^s(0,y)}a_s^1(y)\right],$$

$$\Phi_s(y) = a_{ss}(0,y)[a_s(y) - b_s(0)] + a_s(y), \tag{57}$$

$$\Phi_s(0) = a_{ss}(0,0)a_s(0) + a_s^1(0), \qquad a_s(0) = b_s(0).$$

Substituting the obtained values $\Psi_s(x), \Phi_s(y)$ in system (48) and using the conditions (50) of problem A_3 we get the solution of Volterra system integral equation of the second type which is solvable and its uniqueness solution can be get by using the kernels (52), (53), (54).
The proof of the following Theorem is completed.

Theorem 6 *Let the coefficients of system (1) satisfy: the conditions of Theorem 5, $a_{ss}(0,y) \in C^2(\Gamma_2), a_s(y) \in C^2(\Gamma_2)$ and $b_s(x) \in C^1(|\Gamma_1)$. Then problem A_3 has the unique solution in the form:*

$$U_s(x,y) = g(x,y) + \int_y^\infty \int_x^\infty N_1(x,y;t,\tau)g(t,\tau)dt.d\tau + \int_y^\infty N_2(x,y;\tau)g(x,\tau)d\tau + \\ + \int_y^\infty N_3(x,y;t)g(t,y)dt - \int_y^\infty N_4(x,y;\tau)g(0,\tau)d\tau - \int_x^\infty N_5(x,y;\tau)g(t,0)dt \tag{58}$$

where $g(t,y), U(x,y), N_1, N_2, N_3, N_4, N_5$ are the kernels of the system integral equation (51), (55). The formula (58) of the functions $G_s(x,y)$ can be obtained by using the inequalities (48), (49), (57), (50).

Problem A_4. Find a solution of system (1) within the class $C(D\cup\Gamma_1\cup\Gamma_2)\cap C^2(D)$ with the boundary conditions:

$$\frac{\partial U_s}{\partial y}|_{x=0} = f_s(y), \quad U_s(x,0) = g_s(x). \tag{59}$$

where $f_s(y), g_s(x)$ are continuous functions on Γ_2, Γ_1.
Solution of Problem A_4. From equation (47), we can get the values

$$U_s(x,0) = \Phi_s(x) = g_s(x), \Psi_s(x) = g_s(x)$$

and

$$\Phi_s^1(y) - a_{ss}(0,y)\int_y^\infty \Phi_s^1(\tau)d\tau = F_s(y), \tag{60}$$

By solving equation (60) and substituting the obtained values $\Psi_s(x), \Phi_s(y)$ in equation (48) and then solving the obtained system, we get the solution of Problem A_4.

Theorem 7 *Let in system (1) the coefficients $a(x,y), b(x,y), c(x,y), f(x,y)$ satisfy the conditions of Theorem 5 and in Problem A_4: $f_s(y) \in C^1(\Gamma_1), g_s(x) \in C^2(\Gamma_1)$. Then problem A_4 has the unique solution which is given by the formulae (58), (48), and*

$$\Phi_s(y) = f_s(y) - a_{ss}(0,y).g_s(0)e^{-w_2^s(0,y)} + a_{ss}(0,y).\int_y^\infty \left[\frac{f_s(\tau)}{a_{ss}(0,\tau)} - e^{-w_2^s(0,y)}g_s(0)\right]d\tau. \tag{61}$$

Remak 3 *Similarly, we can investigate system (1) where the coefficients satisfy the conditions* $b_{ss}(x, y) \in C^1_y(D), a_{ss}(x, y), c_s(x, y) \in C(D), s = 1, 2, \ldots, n;$ $a_{js}(x, y) \in C^1_x(D), b_{js}(x, y) = inC^1_y(D)$ *at* $j \neq s, j, s = 1, 2, \ldots, n.$ *Also, we can get a series of new integral representations and a number of boundary value problems.*

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