SOME PROPERTIES OF THE SCHURER TYPE OPERATORS

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Abstract. In this paper are presented first the operators of Schurer which have been introduced and investigated by F. Schurer [11] in 1962 and theirs properties. Then, we study a generalization of Durrmeyer type and a generalization of Kantorovich type the operators of Schurer and we estimate the values of this operators for the test functions. By means of the modulus of continuity of the function used one gives evaluations of the orders of approximation by the considered operators.

1. Introduction

Let $p \in N$ be fixed. In 1962, F. Schurer [11], introduced and investigated the linear positive operator $B_{m,p} : C([0,1+p]) \to C([0,1])$, defined for any $m \in N$ and any $f \in C([0,1+p])$ by

$$(B_{m,p})(x) = \sum_{k=0}^{m+p} \begin{pmatrix} m+p \\ k \end{pmatrix} x^k (1-x)^{m+p-k} f\left(\frac{k}{m}\right),$$

where $B_{m,p}$ are the operators of Bernstein-Schurer. One observe that for $p = 0, B_{m,0}$ we obtain the operators of Bernstein B_m.

Theorem. 1.1. *The operators of Bernstein-Schurer have the following properties:*

i) $(B_{m,p}e_0)(x) = 1, (B_{m,p}e_1)(x) = \left(1 + \frac{p}{m}\right) x$,

$(B_{m,p}e_1)(x) = \frac{m+p}{m^2}\left[(m+p)x^2 + x(1-x)\right]$;

ii) $\lim\limits_{n\to\infty} B_{m,p}f = f$ *uniformly on* $[0,1]$, *for any* $f \in C([0,1+p])$,

iii) $|(B_{m,p}f)(x) - f(x)| \leq 2\omega(f;\delta_{m,p,x})$, *for any* $f \in C([0,1+p])$ *and any* $x \in [0,1]$;

iv) $|(B_{m,p}f)(x) - f(x)| \leq \frac{p}{m}x|f'(x)| + 2\delta_{m,p,x}\omega(f';\delta_{m,p,x})$, *for any* $f \in C^1([0,1+p])$ *and any* $x \in [0,1]$, *if we noticed* $\delta_{m,p,x} = \frac{\sqrt{p^2x^2+(m+p)x(1-x)}}{m}$.

2. A generalization of Durrmeyer type for the operators of Schurer

We consider the operators of Schurer modified into integral form [2] $B^{**}_{m,p}$: $C([0,1+p]) \to C([0,1])$, defined for any $m \in N$ and any $f \in C([0,1+p])$ by

$$\left(B^{**}_{m,p}f\right)(x) = (m+p+1)\sum_{k=0}^{m+p} q^k_{m,p}(x)\int_0^1 q^k_{m,p}(t)\, f(t)\, dt, \qquad (2.1)$$

where

$$q^k_{m,p}(x) = \binom{m+p}{k} x^k (1-x)^{m+p-k}$$

are the fundamental Schurer polynomials.

Theorem. 2.1. *The operators defined by (2.1) have the properties:*

i) $\left(B^{**}_{m,p}e_0\right)(x) = 1$;

ii) $\left(B^{**}_{m,p}e_1\right)(x) = \frac{(m+p)x+1}{m+p+2}$;

iii) $\left(B^{**}_{m,p}e_2\right)(x) = \frac{(m+p)(m+p-1)x^2+4(m+p)x+2}{(m+p+2)(m+p+3)}$.

Proof.

i)

$$\left(B^{**}_{m,p}e_0\right)(x) = (m+p+1)\sum_{k=0}^{m+p}\binom{m+p}{k} x^k (1-x)^{m+p-k}\frac{1}{m+p+1} = 1$$

if on used the relations:

$$\sum_{k=0}^{m+p}\binom{m+p}{k} x^k (1-x)^{m+p-k} = 1$$

and

$$\int_0^1 \binom{m+p}{k} t^k (1-t)^{m+p-k}\, dt = \binom{m+p}{k}\beta(k+1, m+p-k+1)$$

$$= \frac{1}{m+p+1}$$

ii) We have

$$\left(B^{**}_{m,p}e_1\right)(x) = (m+p+1)\sum_{k=0}^{m+p}\binom{m+p}{k}x^k(1-x)^{m+p-k}$$

$$\beta(k+2, m+p-k+1) = (m+p+1)\sum_{k=0}^{m+p}\binom{m+p}{k}x^k(1-k)^{m+p-k}\cdot$$

$$\cdot\frac{k+1}{(m+p+2)(m+p+1)}$$

$$= \frac{1}{m+p+2} + \frac{m+p}{m+p+2}x\sum_{k=1}^{m+p}\binom{m+p-1}{k-1}x^{k-1}(1-x)^{m+p-k}$$

$$= \frac{(m+p)x+1}{m+p+2}.$$

iii) We find

$$\left(B^{**}_{m,p}e_1\right)(x) = (m+p+1)\sum_{k=0}^{m+p}\binom{m+p}{k}x^k(1-x)^{m+p-k}\cdot\beta(k+3, m+p-k+1)$$

$$= (m+p+1)\sum_{k=0}^{m+p}\binom{m+p}{k}x^k(1-x)^{m+p-k}\cdot\frac{(k+2)(k+1)}{(m+p+3)(m+p+2)(m+p+1)}$$

$$= \frac{(m+p)(m+p-1)x^2+4(m+p)x+2}{(m+p+2)(m+p+3)}.$$

Theorem. 2.2. *The operators defined (2.1) have the properties:*
i) $\lim\limits_{n\to\infty} B^{**}_{m,p}f = f$ *uniformly on* $[0,1]$, $(\forall)\, f \in C([0,1+p])$,
ii) $\left|\left(B^{**}_{m,p}f\right)(x) - f(x)\right| \le 2\omega\left(f; \frac{1}{\sqrt{2(m+p+3)}}\right)$, $(\forall)\, f \in C([0,1+p])$,$(\forall)\, x \in [0,1]$, $m \ge 3$, $p \in N$ *is fixed.*

Proof.
i) It results from Bohman-Korovkin theorem
ii) We used the properties:

If L is a linear positive operator L: $C(I) \to C(I)$, such that $Le_0 = e_0$ then $|(Lf)(x) - f(x)| \leq \left(1 + \delta^{-1}\sqrt{(L\varphi_x^2)(x)}\right)\omega(f;\delta)$, $(\forall) f \in C_B(I)$, $(\forall) x \in I$, $\delta\rangle 0$ and $\varphi_x = |t - x|$.

We have

$$\left|\left(B_{m,p}^{**}f\right)(x) - f(x)\right| \leq \left(1 + \delta^{-1}\sqrt{B_{m,p}^{**}\varphi_x^2}\right)\omega(f;\delta),$$

$$\left(B_{m,p}^{**}\varphi_x^2\right)(x) = \left(B_{m,p}^{**}e_2\right)(x) - 2x\left(B_{m,p}^{**}e_1\right)(x) + x^2\left(B_{m,p}^{**}e_0\right)(x)$$

$$= \frac{2(m+p-3)x(1-x)+2}{(m+p+2)(m+p+3)}.$$

If $m + p \geq 3$ it is maximal for x $= \frac{1}{2}$ and we find

$$\left(B_{m,p}^{**}\varphi_x^2\right)(x) \leq \frac{m+p+1}{2(m+p+2)(m+p+3)}.$$

We get

$$\left|\left(B_{m,p}^{**}f\right)(x) - f(x)\right| \leq \left(1 + \delta^{-1}\sqrt{\frac{m+p+1}{2(m+p+2(m+p+3)}}\right)\omega(f;\delta)$$

$$\leq \left(1 + \delta^{-1}\sqrt{\frac{1}{2(m+p+3)}}\right)\omega(f;\delta).$$

For $\delta = \frac{1}{\sqrt{2(m+p+3)}}$ we obtain the inequalities

$$\left|\left(B_{m,p}^{**}f\right)(x) - f(x)\right| \leq 2\omega\left(f;\frac{1}{\sqrt{2(m+p+3)}}\right).$$

3. A generalizations of Kantorovich type for the operators of Schurer

We consider the operators of Schurer modified into integral form [3]

$$B_{m,p}^{*}: C([0,1+p]) \to C([0,1]),$$

defined for any $f \in C([0,1+p])$ and any $x \in [0,1]$ by

$$\left(B^*_{m,p}f\right)(x) = (m+p+1)\sum_{k=0}^{m+p}\left(\begin{array}{c} m+p \\ k \end{array}\right) x^k (1-x)^{m+p-k} \int\limits_{\frac{k}{m+p+1}}^{\frac{k+1}{m+p+1}} f(t)\,dt \tag{3.1}$$

Theorem. 3.1. *The operators defined by (3.1) have the properties:*
i) $\left(B^*_{m,p}e_0\right)(x) = 1$;
ii) $\left(B^*_{m,p}e_1\right)(x) = \frac{m+p}{m+p+}x + \frac{1}{2(m+p+1)}$;
iii) $\left(B^*_{m,p}e_2\right)(x) = \frac{(m+p)(m+p-1)}{(m+p+1)^2}x^2 + \frac{2(m+p)}{(m+p+1)^2}x + \frac{1}{3(m+p+1)^2}$.
Proof.
i) $\left(B^*_{m,p}e_0\right)(x) = (m+p+1)\sum\limits_{k=0}^{m+p}\left(\begin{array}{c} m+p \\ k \end{array}\right) x^k(1-x)^{m+p-k} t \Big|_{\frac{k}{m+p+1}}^{\frac{k+1}{m+p+1}} =$
$= (m+p+1)\sum\limits_{k=0}^{m+p}\left(\begin{array}{c} m+p \\ k \end{array}\right) x^k(1-x)^{m+p-k}\frac{1}{m+p+1} = 1$, if on used the relation:

$$\sum_{k=0}^{m+p}\left(\begin{array}{c} m+p \\ k \end{array}\right) x^k(1-x)^{m+p-k} = 1.$$

ii) We have
$\left(B^*_{m,p}e_1\right)(x) = (m+p+1)\sum\limits_{k=0}^{m+p}\left(\begin{array}{c} m+p \\ k \end{array}\right) x^k(1-x)^{m+p-k}\frac{t^2}{2} \Big|_{\frac{k}{m+p+1}}^{\frac{k+1}{m+p+1}} =$
$= \frac{1}{2(m+p+1)}\sum\limits_{k=0}^{m+p}\left(\begin{array}{c} m+p \\ k \end{array}\right) x^k(1-x)^{m+p-k}(2k+1) = \frac{m+p}{m+p+1}x + \frac{1}{2(m+p+1)}$.
iii) We find
$\left(B^*_{m,p}e_2\right)(x) = (m+p+1)\sum\limits_{k=0}^{m+p}\left(\begin{array}{c} m+p \\ k \end{array}\right) x^k(1-x)^{m+p-k}\frac{t^3}{3} \Big|_{\frac{k}{m+p+1}}^{\frac{k+1}{m+p+1}} =$
$= \frac{1}{3(m+p+1)}\sum\limits_{k=0}^{m+p}\left(\begin{array}{c} m+p \\ k \end{array}\right) x^k(1-x)^{m+p-k}(3k^2+3k+1) =$
$= \frac{(m+p)(m+p+1)}{(m+p+1)^2}x^2 + \frac{2(m+p)}{(m+p+1)^2}x + \frac{1}{3(m+p+1)^2}$.

Theorem. 3.2. *The operators defined by (3.1) have the properties:*
i) $\lim\limits_{m\to\infty}\left(B^*_{m,p}f\right)(x) = f(x)$ *, uniformly on* $[0,1]$, $(\forall)\, f \in C([0,1+p])$ *;*
ii) $\left|\left(B^*_{m,p}f\right)(x) - f(x)\right| \le 2\omega\left(f; \frac{1}{2\sqrt{m+p+1}}\right)$, $(\forall)\, f \in C([0,1+p])$, $(\forall)\, x \in [0,1]$ *and* $p \in N$ *is fixed.*
Proof.

i) It results from Bohman-Korovkin theorem

ii) We used the properties:

If L is a linear positive operator L: $C(I) \to C(I)$, such that $Le_0 = e_0$ then $|(Lf)(x) - f(x)| \le \left(1 + \delta^{-1}\sqrt{(L\varphi_x^2)(x)}\right)\omega(f;\delta)$, $(\forall) f \in C_B(I)$, $(\forall) x \in I$, $\delta\rangle 0$ and $\varphi_x = |t - x|$.

We have

$$\left|\left(B_{m,p}^* f\right)(x) - f(x)\right| \le \left(1 + \delta^{-1}\sqrt{B_{m,p}^*\varphi_x^2}\right)\omega(f;\delta),$$

$$\left(B_{m,p}^*\varphi_x^2\right)(x) = \frac{m+p-1}{(m+p+1)^2}x(1-x) + \frac{1}{3(m+p+1)^2}.$$

For $\delta < \frac{1}{2\sqrt{m+p+1}}$, we find the inequality:

$$\left|\left(B_{m,p}^* f\right)(x) - f(x)\right| \le 2\omega\left(f;\frac{1}{\sqrt{2(m+p+1)}}\right).$$

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