THE FIXED POINT PROPERTY IN CONVEX MULTI-OBJECTIVE OPTIMIZATION PROBLEM

Zdravko Dimitrov Slavov

Abstract. In this paper we study the Pareto-optimal solutions in convex multi-objective optimization with compact and convex feasible domain. One of the most important problems in multi-objective optimization is the investigation of the topological structure of the Pareto sets. We present the problem of construction of a retraction function of the feasible domain onto Pareto-optimal set, if the objective functions are concave and one of them is strictly quasi-concave on compact and convex feasible domain. Using this result it is also proved that the Pareto-optimal and Pareto-front sets are homeomorphic and they have the fixed point property.

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1. Introduction

The key idea of the present paper is first to show how we can construct a retraction of the feasible domain onto Pareto-optimal set in multi-objective optimization problem. Next, using this function we will prove that the Pareto-optimal and Pareto-front sets are homeomorphic and they have the fixed point property.

In a general form, the multi-objective optimization problem $MOP(X, F)$ is to find $x \in X \subset R^m$, $m \geq 1$, so as to maximize $F(x) = (f_1(x), f_2(x), ..., f_n(x))$ subject to $x \in X$, where the feasible domain X is nonempty, convex and compact, $J = \{1, 2, ..., n\}$ is the index set, $n \geq 2$, $f_i : X \to R$ is given continuous objective function for all $i \in J$.

Definitions of the Pareto-optimal solutions can be formally stated as follows:

(a) A point $x \in X$ is called Pareto-optimal solution if and only if there does not exist a point $y \in X$ such that $f_i(y) \geq f_i(x)$ for all $i \in J$ and $f_k(y) > f_k(x)$ for some $k \in J$. Denote the set of the Pareto-optimal solutions of X by $Max(X, F)$ and it is called Pareto-optimal set. The set $F(Max(X, F)) = Eff(F(X))$ is called Pareto-front set or efficient set.

(b) A point $x \in X$ is called weakly Pareto-optimal solution if and only if there does not exist a point $y \in X$ such that $f_i(y) > f_i(x)$ for all $i \in J$. Denote the set of the weakly Pareto-optimal solutions of X by $WMax(X, F)$ and it is called weakly Pareto-optimal set. The set $F(WMax(X, F)) = WEff(F(X))$ is called weakly Pareto-front set or weakly efficient set.

One of the most important problems of optimization problem $MOP(X, F)$ is the investigation of the structure of the Pareto-optimal set $Max(X, F)$ and the Pareto-front set $Eff(F(X))$, see also [7] and [11]. Considering topological properties of the efficient set is started by [10].

As it is well-known the Pareto-optimal set $Max(X, F)$ is nonempty, the weakly Pareto-optimal set $WMax(X, F)$ is nonempty and compact, $Max(X, F) \subset WMax(X, F)$ and $Eff(F(X)) = WEff(F(X))$. It can be shown that both sets $Eff(F(X))$ and $WEff(F(X))$ lie in the boundary of the set $F(X)$, i.e. $F(Max(X, F)) \subset \partial F(X)$ and $F(WMax(X, F)) \subset \partial F(X)$.

If the functions $\{f_i\}_{i=1}^{n}$ are strictly quasi-concave on X, then $Max(X, F) = WMax(X, F)$ [7]. Therefore, under these assumptions the Pareto-optimal set $Max(X, F)$ is compact.

Topological properties of the Pareto solutions sets (Pareto-optimal and Pareto-front) in multi-objective optimization have been discussed by several authors. Connectedness and path-connectedness are considered in [1], [8], [12], [13] and [15]. In [2], it is proved that the efficient set in strictly quasi-concave multi-objective optimization with compact feasible domain is contractible. In [5], it is proved that the Pareto solutions sets in strictly quasi-concave multi-objective optimization are contractible, if any intersection of level sets of the objective functions with the feasible domain is a compact set.

In this paper, let the functions $\{f_i\}_{i=1}^{n}$ be concave and a function f_λ of $\{f_i\}_{i=1}^{n}$ be strictly quasi-concave on the convex domain X. The central aim is to:

(1) construct a retraction $r : X \to Max(X, F)$.

(2) prove that $Max(X, F)$ and $Eff(F(X))$ are homeomorphic and have

the fixed point property.

2. General definitions and notions

We will use R^m and R^n as the genetic finite-dimensional vector spaces.

In addition, we also introduce the following notations: for every two vectors $x, y \in R^n$, $x(x_1, x_2, ..., x_n) \geq y(y_1, y_2, ..., y_n)$ means $x_i \geq y_i$ for all $i \in J$ (weakly componentwise order), $x(x_1, x_2, ..., x_n) > y(y_1, y_2, ..., y_n)$ means $x_i > y_i$ for all $i \in J$ (strictly componentwise order), and $x(x_1, x_2, ..., x_n) \succeq y(y_1, y_2, ..., y_n)$ means $x_i \geq y_i$ for all $i \in J$ and $x_k > y_k$ for some $k \in J$ or $x \geq y$ and $x \neq y$ (componentwise order).

We will use the definitions of concave, quasi-concave and strictly quasi-concave function in the usual sense:

(a) A function f is concave on X if and only if for any $x, y \in X$ and $t \in [0, 1]$, then $f(tx + (1-t)y) \geq tf(x) + (1-t)f(y)$.

(b) A function f is quasi-concave on X if and only if for any $x, y \in X$ and $t \in [0, 1]$, then $f(tx + (1-t)y) \geq min(f(x), f(y))$.

(c) A function f is strictly quasi-concave on X if and only if for any $x, y \in X$, $x \neq y$ and $t \in (0, 1)$, then $f(tx + (1-t)y) > min(f(x), f(y))$.

Let a function $dis : X \times X \to R_+$ be a metric (or distance) in X. In a metric space (X, dis), let τ be a topology induced by dis. In a topological space (X, τ), for set $Y \subset X$ we recall some definitions:

(a) The set Y is called connected if and only if it is not the union of a pair of nonempty sets of τ, which are disjoint.

(b) The set Y is called path-connected (arc-connected or arcwise-connected) if and only if for every $x, y \in Y$ there exists a continuous function $p : [0, 1] \to Y$ such that $p(0) = x$ and $p(1) = y$. The function p is called path.

(c) The set Y is a retract of X (or X is a retract to Y) if and only if there exists a continuous function $r : X \to Y$ such that $r(X) = Y$ and $r(x) = x$ for all $x \in Y$. The function r is called retraction of X to Y.

(d) A continuous function $d : X \times [0, 1] \to X$ is a deformation retraction of X onto Y if and only if $d(x, 0) = x$, $d(x, 1) \in Y$ and $d(a, t) = a$ for all $x \in X$, $a \in Y$ and $t \in [0, 1]$. The set Y is called a deformation retract of X.

(e) The set Y is contractible if and only if there exist a continuous function $c : Y \times [0, 1] \to Y$ and $a \in Y$ such that $c(x, 0) = a$ and $c(x, 1) = x$ for all $x \in Y$. In the other words, Y is contractible if there exists a deformation retract of Y onto a point. The function c is called contraction.

(f) The set Y is said to have a fixed point property if and only if every continuous function $f : Y \to Y$ from this set into itself has a fixed point, i.e. there is a point $x \in Y$ such that $x = f(x)$.

Of course, the compactness, connectedness and path-connectedness of the Pareto-optimal set are related to the compactness, connectedness and path-connectedness of the Pareto-front set, respectively.

From a more formal viewpoint, a retraction is a point-to-point mapping $r : X \to Y$ that fixes every point of Y and $r \circ r(x) = r(x)$ for all $x \in X$. Retractions are the topological analog of projection operators in other parts on mathematics.

It is clear to see that every deformation retraction is a retraction, $r(x) = d(x, 1)$ for all $x \in X$. But in generally the converse does not hold [4].

The fixed point property of sets are preserved under retractions. This means that the following statement is true: If the set X has the fixed point property and Y is a retract of X, then the set Y has the fixed point property.

Let X and Y be topological spaces and let $h : X \to Y$ be bijective. Then h is homeomorphism if and only if h and h^{-1} are continuous. If such a homeomorphism h exists, then X and Y are called homeomorphic (or X is homeomorphic to Y). A property of topological spaces which when possessed by a spaces is also possessed by every spaces homeomorphic to it is called a topological property or a topological invariant. The fixed point property of sets are preserved under homeomorphisms.

3. Main result

Now, under our assumptions, the functions $\{f_i\}_{i=1}^n$ are concave and the function f_λ of $\{f_i\}_{i=1}^n$ is strictly quasi-concave on the convex domain X, we will construct the retraction and discuss some topological properties of the Pareto solutions sets.

To begin with the following definitions:

(a) Define a function $f : X \to R$ by $f(x) = \Sigma_{i=1}^n f_i(x)$ for all $x \in X$. It is clear to check that the function f is concave on X and $Argmax(f, X) \subset Max(X, F)$.

(b) Define also a point-to-set mapping $\rho : X \rightrightarrows X$ by $\rho(x) = \{y \in X \mid F(y) \geq F(x)\}$. It can be shown that the set $\rho(x)$ is a nonempty, convex and compact set for all $x \in X$ and there is $x \in \rho(x)$. Hence, the point-to-set mapping ρ is convex-valued and compact-valued on X.

These definitions allow us to present a main theorem of our paper.

Theorem 1. *There exists a retraction* $r : X \to Max(X,F)$ *such that* $r(X) = Max(X,F)$ *and* $r(x) = Argmax(f, \rho(x))$ *for all* $x \in X$.

In order to give the prove of Theorem 1, we will construct the retraction r. The idea is to transfer the multi-objective optimization problem to mono-objective optimization problem by define a unique objective function.

Now, let fix an arbitrary point $x \in X$ and denote $t_i = f_i(x)$ for $i \in J$. Consider an optimization problem with single objective function as follows: maximize $f(y)$ subject to $y \in \rho(x)$.

In result, we get an equivalent optimization problem: maximize $f(y)$ subject to $g_i(y) \geq 0, i \in J$ and $y \in X$, where the functions $g_i : X \to R$ satisfying $g_i(y) = f_i(y) - t_i$ for $i \in J$. Note that the objective function f and the constraint functions $\{g_i\}_{i=1}^n$ are all concave on the convex domain X, see [3].

We will show that these problems have a unique solution $x^* \in Max(X,F)$. Thus, a retraction $x^* = r(x)$ will be constructed.

At first, we will prove some lemmas.

Lemma 1. *If* $x \in X$, *then* $\mid Argmax(f, \rho(x)) \mid= 1$ *and* $Argmax(f, \rho(x)) \subset Max(X,F)$.

Proof. Clearly, there is $\mid Argmax(f, \rho(x)) \mid\geq 1$. Let choose $y_1, y_2 \in Argmax(f, \rho(x))$, $y_1 \neq y_2$, $t \in (0,1)$ and $z = ty_1 + (1-t)y_2$. It is known that the set $Argmax(f, \rho(x))$ is convex, therefore there is $z \in Argmax(f, \rho(x))$. Thus, we obtain $f(z) = f(y_1) = f(y_2)$.

For each $i \in J$ there is $f_i(z) \geq tf_i(y_1) + (1-t)f_i(y_2)$. By using this result we derive that $f(z) \geq tf(y_1) + (1-t)f(y_2) = f(y_1) = f(y_2)$. Since $f(z) = f(y_1) = f(y_2)$ implies $f_i(z) = tf_i(y_1) + (1-t)f_i(y_2)$ for all $i \in J$ and for all $t \in (0,1)$. As a result, we get that $f_i(z) = f_i(y_2) + t(f_i(y_1) - f_i(y_2))$ for all $t \in (0,1)$, therefore we find that $f_i(y_1) = f_i(y_2)$ for all $i \in J$.

Now, let fix $t \in (0,1)$. As described above, the function f_λ is strictly quasi-concave, therefore we obtain $f_\lambda(z) > min(f_\lambda(y_1), f_\lambda(y_2)) = f_\lambda(y_1) = f_\lambda(y_2)$. But $f_i(z) \geq tf_i(y_1) + (1-t)f_i(y_2)$ for all $i \in J$ and by using this result we derive that $f(z) > tf(y_1) + (1-t)f(y_2) = f(y_1)$. This leads to a contradiction, therefore we obtain $\mid Argmax(f, \rho(x)) \mid= 1$.

Let choose an arbitrary point $y \in Argmax(f, \rho(x))$ and assume that $y \notin Max(X,F)$. From the assumption $y \notin Max(X,F)$ it follows that there exists $z \in X$ satisfying $F(x) \succeq F(y)$. As a result we derive that $z \in \rho(x)$ and $f(z) > f(y)$. Again, this leads to a contradiction, therefore we obtain $y \in Max(X,F)$.

The lemma is proved.

Thus, we introduced the idea of the retraction.

Now, using the results of Lemma 1 we are in position to construct a function $r : X \to Max(X, F)$ such that $r(x) = Argmax(f, \rho(x))$ for all $x \in X$.

Lemma 2. *If $x \in X$, $x \in Max(X, F)$ is equivalent to $\rho(x) = \{x\}$.*

Proof. Let $x \in Max(X, F)$ and assume that $\rho(x) \neq \{x\}$. From both conditions $x \in \rho(x)$ and $\rho(x) \neq \{x\}$ it follows that there exists $y \in \rho(x) \setminus \{x\}$ such that $F(y) \geq F(x)$. Let choose $t \in (0, 1)$ and $z = tx + (1 - t)y$ therefore $z \in \rho(x)$. Since $x \neq y$ implies $f_\lambda(z) > f_\lambda(x)$, which contradicts condition $x \in Max(X, F)$ therefore we obtain $\rho(x) = \{x\}$.

Conversely, let $\rho(x) = \{x\}$ and assume that $x \notin Max(X, F)$. From the assumption $x \notin Max(X, F)$ it follows that there exists $y \in X$ satisfying $F(y) \succeq F(x)$. Thus we deduce that $y \in \rho(x)$ and $x \neq y$, which contradicts condition $\rho(x) = \{x\}$ therefore we obtain $x \in Max(X, F)$.

The lemma is proved.

Applying now the previous lemma it follows that if $x \in Max(X, F)$, then $x = r(x)$ and if $x \notin Max(X, F)$, then $x \neq x^* = r(x)$. It is easy verify direct that $r \circ r = r$.

Lemma 3. $r(X) = Max(X, F)$.

Proof. From Lemmas 1 it follows that $r(X) \subset Max(X, F)$. Applying Lemma 2 we deduce $r(Max(X, F)) = Max(X, F)$. This means that $r(X) = Max(X, F)$.

The lemma is proved.

We will analyze the point-to-set mapping ρ. Using the Maximum Theorem, one of the fundamental results of optimization theory, we will show that the function r is continuous.

Lemma 4. *If $\{x_k\}_{k=1}^{\infty}$, $\{y_k\}_{k=1}^{\infty} \subset X$ are pair of sequences such that $lim_{k\to\infty} x_k = x_0 \in X$ and $y_k \in \rho(x_k)$ for all $k \in N$, then there exists a convergent subsequence of $\{y_k\}_{k=1}^{\infty}$ whose limit belongs to $\rho(x_0)$.*

Proof. Since the assumption $y_k \in \rho(x_k)$ for all $k \in N$ implies $f_i(y_k) \geq f_i(x_k)$ for all $k \in N$ and all $i \in J$. From the condition $\{y_k\}_{k=1}^{\infty} \subset X$ it follows that there exists a convergent subsequence $\{q_k\}_{k=1}^{\infty} \subset \{y_k\}_{k=1}^{\infty}$ such that $lim_{k\to\infty} q_k = y_0 \in X$. Therefore, there exists a convergent subsequence $\{p_k\}_{k=1}^{\infty} \subset \{x_k\}_{k=1}^{\infty}$ such that $q_k \in \rho(p_k)$ and $lim_{k\to\infty} p_k = x_0$. Thus, we find that $f_i(q_k) \geq f_i(p_k)$ for all $k \in N$ and for all $i \in J$. Taking the limit as $k \to \infty$ we obtain $f_i(y_0) \geq f_i(x_0)$ for all $i \in J$. This implies $y_0 \in \rho(x_0)$.

The lemma is proved.

Continuing with this analysis, we have the following lemma.

Lemma 5. *If* $\{x_k\}_{k=1}^{\infty} \subset X$ *is a convergent sequence to* $x_0 \in X$ *and* $y_0 \in \rho(x_0)$*, then there exists a sequence* $\{y_k\}_{k=1}^{\infty} \subset X$ *such that* $y_k \in \rho(x_k)$ *for all* $k \in N$ *and* $lim_{k\to\infty} y_k = y_0$*.*

Proof. Let denote a distance between a point y_0 and a set $\rho(x_k)$ by $d_k = inf\{dis(y_0, x) \mid x \in \rho(x_k)\}$. As already noted, $\rho(x_k)$ is a nonempty, convex and compact set. Observe that if $y_0 \notin \rho(x_k)$, then there exists a unique $y^* \in \rho(x_k)$ such that $d_k = d(y^*, y_k)$.

There are two cases as follows:

First, if $y_0 \in \rho(x_k)$, then $d_k = 0$ and let $y_k = y_0$.

Second, if $y_0 \notin \rho(x_k)$, then $d_k > 0$ and let $y_k = y^*$.

As a result, we get a sequence $\{d_k\}_{k=1}^{\infty} \subset R_+$ and a sequence $\{y_k\}_{k=1}^{\infty} \subset X$ such that $y_k \in \rho(x_k)$ for all $k \in N$ and $d_k = dis(y_0, y_k)$. Since $lim_{k\to\infty} x_k = x_0$ implies the sequence $\{d_k\}_{k=1}^{\infty}$ is convergent and $lim_{k\to\infty} d_k = 0$. Finally, we obtain $lim_{k\to\infty} y_k = y_0$.

The lemma is proved.

Lemma 6. *The point-to-set mapping* ρ *is continuous on* X*.*

Proof. On one hand, from Lemma 4 it follows that the point-to-set mapping ρ is upper semi-continuous on X [9]. On the other hand, from Lemma 5 it follows that the point-to-set mapping ρ is lower semi-continuous on X [9]. This shows that the point-to-set mapping ρ is continuous on X.

The lemma is proved.

Lemma 7 [14, Theorem 9.14 - The Maximum Theorem]. *Let* $S \subset R^n$, $\Theta \subset R^m$, $g : S \times \Theta \to R$ *a continuous function, and* $D : \Theta \rightrightarrows S$ *be a compact-valued and continuous point-to-set mapping. Then, the function* $g^* : \Theta \to R$ *defined by* $g^*(\theta) = max\{g(x, \theta) \mid x \in D(\theta)\}$ *is continuous on* Θ*, and the point-to-set mapping* $D^* : \Theta \rightrightarrows S$ *defined by* $D^*(\theta) = \{x \in D(\theta) \mid g(x, \theta) = g^*(\theta)\}$ *is compact-valued and upper semi-continuous on* Θ*.*

Lemma 8. *The function* r *is continuous on* X*.*

Proof. Let apply Lemma 7 for $X = S = \Theta$. Obviously, the function f is continuous on X. As mentioned before, the point-to-set mapping ρ is compact-valued and continuous on X. According to Lemma 1, from the fact $\mid Argmax(f, \rho(x)) \mid = 1$, we deduce that r is upper semi-continuous point-to-point mapping. As it is well-known that every point-to-point mapping, that is upper semi-continuous, is continuous when viewed as a function. In result, the function r is continuous on X.

The lemma is proved.

We are now in the position to prove the main result of this section.

Proof of Theorem 1. From Lemmas 1, 3 and 8 it follows that there exists a continuous function $r : X \to Max(X, F)$ such that $r(X) = Max(X, F)$ and $r(x) = Argmax(f, \rho(x))$ for all $x \in X$.

This completed the proof of our theorem.

To recall that a property P is called a topological property if and only if an arbitrary set X has this property, then Y has this property too, where X and Y are homeomerphic.

Theorem 2. $Max(x, F)$ is homeomorphic to $Eff(F(X))$.

Proof. As it is well-known every continuous image of the compact set is compact. In fact, the set X is compact and the function r is continuous on X. Therefore, the set $Max(X, F) = r(X)$ is compact.

Since the function $F : X \to R^n$ is continuous it follows that the restriction $h : Max(X, F) \to F(Max(X, F))$ of F is continuous too. Applying Lemma 2 we deduce that if $x, y \in Max(X, F)$ and $x \neq y$, then $h(x) \neq h(y)$. We derive that the function h is bijective. Consider the inverse function $h^{-1} : F(Max(X, F)) \to Max(X, F)$ of h. As proved before, the set $Max(x, F)$ is compact, therefore h^{-1} is continuous too. Finally, we obtain that the function h is homeomorphism.

This completed the proof of our theorem.

The fixed point property is related to the notion of retraction. As showed before, if X has the fixed point property and Y is a retract of X, then Y also has fixed point property.

Theorem 3. $Max(X, F)$ and $Eff(F(X))$ *have the fixed point property.*

In the proof of this theorem, we will use the following lemmas.

Lemma 10 [14, Theorem 9.31 - Schauder's Fixed Point Theorem]. *Let $f : S \to S$ be continuous function from nonempty, compact and convex set $S \subset R^n$ into itself, then f has a fixed point.*

Lemma 11. $Max(X, F)$ *has a fixed point property.*

Proof. In fact, the set X is nonempty, compact and convex. Hence, from Lemma 10 implies that it has the fixed point property. As we have shown in Theorem 1, the set $Max(X, F)$ is a retract of X. As described earlier, the fixed point property is preserved under retraction. Then, the set $Max(X, F)$ has the fixed point property.

The lemma is proved.

Proof of Theorem 3. As we have proved in Lemma 11, the set $Max(X, F)$ has the fixed point property. As mentioned before, the fixed point property is preserved under homeomorphism. Now, applying Theorem 2 we obtain that the set $Eff(F(X))$ has the fixed point property too.

This completed the proof of our theorem.

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Author:

Zdravko Dimitrov Slavov
Department of Mathematics, Varna Free University,
Golden Sands Resort, Varna 9007, Bulgaria
e-mail:*slavovibz@yahoo.com*