SOME QUALITATIVE FEATURES OF 2D PERIODIC MIXING MODEL

Adela Ionescu, Mihai Costescu

Abstract. The problems of flow kinematics are far from complete solving. Recently, the mixing theory issued in this field, with the mathematical methods and techniques, developed the significant relation between turbulence and chaos. The turbulence is an important feature of dynamic systems with few freedom degrees, the so-called "far from equilibrium systems", which are widespread between the models of excitable media.

In the previous works, the study of the 3D non-periodic models exhibited a quite complicated behavior. In agreement with experiments, they involved some significant events - the so-called "rare events". The variation of parameters had a great influence on the length and surface deformations.

The 2D (periodic) case is simpler, but significant events can issue for irrational values of the length and surface versors, as is the situation in 3D case. Also, the graphic analysis previously realized involved that in 2D case, the mixing has also a nonlinear behavior and the rare events can appear.

In this paper is started a computational analysis for 2D basic mixing model, in a modified version. In the first stage, there is analyzed the length deformation for different values of the basic parameters. For the simulations there are used specific procedures and functions of MapleVI. The conclusions will be further used for analyzing the mixing efficiency.

2000 Mathematics Subject Classification: 76F25, 37E35

1.Introduction

In the turbulence theory, two important fields are distinguished:

a) The transition theory from smooth laminar flows to chaotic flows, characteristic to turbulence.

b) Statistic studies of the complete turbulent systems.

From statistical standpoint, a flow is represented by the map:

$$x = \Phi_t (X), X = \Phi_t (t = 0) (X) \tag{1}$$

which must be of class C^k. From the dynamic standpoint the map:

$$\Phi_t (X) \longrightarrow x \tag{2}$$

is a diffeomorphism of class C^k and (1) must satisfy the relation

$$0 < J < \infty, J = \det \left(\frac{\partial x_i}{\partial X_j} \right), J = det(D\Phi_t(X)) \tag{3}$$

where D denotes the derivation with respect to the reference configuration, in this case X. The relation (3) implies two particles, X_1 and X_2, which occupy the same position x at a moment. Non-topological behavior (like break up, for example) *is not allowed*.

With respect to X it is defined the basic measure for the deformation, namely the *deformation gradient*, **F**:

$$\mathbf{F} = \left(\nabla_X \Phi_t (\mathbf{X})\right)^T, F_{ij} = \left(\frac{\partial x_i}{\partial X_j} \right), \textit{ or } \mathbf{F} = D\Phi_t (\mathbf{X}) \tag{4}$$

where ∇_X denotes differentiation with respect to X. According to (3), **F** is non singular. The basic measure for the deformation with respect to x is the *velocity gradient* (∇_x denote differentiation with respect to x).

By differentiation of x with respect to X there are obtained the basic deformation for a material filament, and for the area of an infinitesimal material surface [4].

Let us focus on the basic deformation measures:the *length deformation* λ and *surface deformation* η , with the relations [4,5]:

$$\lambda = (C : MM)^{\frac{1}{2}}, \ \eta = (\det F) \cdot \left(C^{-1} : NN\right)^{\frac{1}{2}}, \tag{5}$$

with $\mathbf{C} = \mathbf{F}^T\mathbf{F}$ the Cauchy-Green deformation tensor, and the length and surface vectors M, N defined by

$$\mathbf{M} = d\mathbf{X}/ |d\mathbf{X}|, \ \mathbf{N} = d\mathbf{A}/ |d\mathbf{A}| \tag{6}$$

The relation (6) has the following scalar form:

$$\lambda = C_{ij} \cdot M_i \cdot N_j, \quad \eta = (\det F) \cdot \left(C_{ij}^{-1} \cdot N_i \cdot N_j \right), \tag{7}$$

with $\sum M_i^2 = 1$, $\sum N_j^2 = 1$.

The deformation tensor $\mathbf{F}$ and the associated tensors $\mathbf{C}$, $\mathbf{C}^{-1}$ represent the basic quantities in the deformation analysis for the infinitesimal elements.

In this framework, the mixing concept implies the stretching and folding of the material elements. If in an initial location P there is a material filament dX and an area element dA, the specific length and surface deformations are given by the relations:

$$\frac{D(\ln \lambda)}{Dt} = \mathbf{D} : \mathbf{mm}, \quad \frac{D(\ln \eta)}{Dt} = \nabla \mathbf{v} - \mathbf{D} : \mathbf{nn} \tag{8}$$

where D is the deformation tensor, obtained by decomposing the velocity gradient in its symmetric and non-symmetric part [5]:

$$\nabla \mathbf{v} = \mathbf{D} + \mathbf{\Omega}$$

$$\mathbf{D} = \frac{\left(\nabla \mathbf{v} + (\nabla \mathbf{v})^T \right)}{2} \; \textit{the symmetric tensor} \tag{9}$$

$$\mathbf{\Omega} = \frac{\left(\nabla \mathbf{v} - (\nabla \mathbf{v})^T \right)}{2} \; \textit{the antisimetric tensor}$$

2. The perturbed 2D mixing model. Results

Studying a mixing for a flow implies the analysis of successive *stretching* and *folding* phenomena for its particles, the influence of parameters and initial conditions [6]. In the previous works, the study of the 3D non-periodic models exhibited a quite complicated behavior [1]. For the moment the aim is to study the behavior of the length deformation of the modified 2D mixing model, for some irrational values of the length versor, in order to search some significant events, and compare to 3D case.

Let us start from the basic (widespread) 2D mixing model [5]:

$$\begin{cases} \dot{x}_1 = G \cdot x_2 \\ \dot{x}_2 = K \cdot G \cdot x_1 \end{cases}$$

and consider a little perturbation of it, namely:

$$\begin{cases} \dot{x}_1 = G \cdot x_2 + x_1 \\ \dot{x}_2 = K \cdot G \cdot x_1 + x_2 \end{cases} \tag{10}$$

with $-1 < K < 1$, $0 < G < 1$.

Here is the time derivative. If we attach the initial condition:

$$x_1(0) = X_1, x_2(0) = X_2 \tag{11}$$

the Cauchy problem (10)-(11) has the following calculated solution [3]:

$$x_1 = \left[\frac{X_2}{2} \cdot \frac{P - \sqrt{P}}{KG} - \frac{X_1}{2} \cdot \left(\frac{P}{\sqrt{P}} + 1\right)\right] \cdot \exp\left(1 - \sqrt{P}\right) t +$$

$$\left[\frac{X_1}{2} \cdot \left(\frac{P}{\sqrt{P}} + 1\right) - \frac{X_2}{2} \cdot \frac{(P-1) \cdot \sqrt{P}}{KG}\right] \cdot \exp\left(1 + \sqrt{P}\right) t \tag{12}$$

and

$$x_2 = \left(\frac{X_2}{2} - \frac{X_1}{2} \cdot \frac{KG}{\sqrt{P}}\right) \cdot \exp\left(1 - \sqrt{P}\right) t +$$

$$\left[\frac{X_1}{2} \cdot \frac{KG}{\sqrt{P}} + \frac{X_2}{2} \cdot \left(1 - \sqrt{P}\right)\right] \cdot \exp\left(1 + \sqrt{P}\right). \tag{13}$$

Therefore, the deformation gradient (4) is found as:

$$\mathbf{F} = \begin{pmatrix} \begin{matrix} -\frac{1}{2}\left(\frac{P}{\sqrt{P}} - 1\right) \cdot \exp\left(1 - \sqrt{P}\right) t + \\ \frac{1}{2}\left(\frac{P}{\sqrt{P}} + 1\right) \cdot \exp\left(1 + \sqrt{P}\right) t \end{matrix} & \begin{matrix} \frac{1}{2}\frac{P - \sqrt{P}}{KG} \cdot \exp\left(1 - \sqrt{P}\right) t - \\ \frac{1}{2}\frac{(P-1)\sqrt{P}}{KG} \cdot \exp\left(1 + \sqrt{P}\right) t \end{matrix} \\ \begin{matrix} -\frac{1}{2}\frac{KG}{\sqrt{P}} \cdot \exp\left(1 - \sqrt{P}\right) t + \\ \frac{1}{2}\frac{KG}{\sqrt{P}} \cdot \exp\left(1 + \sqrt{P}\right) t \end{matrix} & \begin{matrix} \frac{1}{2}\exp\left(1 - \sqrt{P}\right) t + \\ \frac{1}{2}\left(1 - \sqrt{P}\right) \cdot \exp\left(1 + \sqrt{P}\right) t \end{matrix} \end{pmatrix} \tag{14}$$

where $2 + KG^2 \overset{not}{=} P$.

The transposed matrix $\mathbf{F}^T$ follows immediately and the Cauchy-Green tensor

$$\mathbf{C} = \mathbf{F}^T \cdot \mathbf{F}$$

has the classical form,

$$\mathbf{C} = \begin{pmatrix} c_{11} & c_{12} \\ c_{21} & c_{22} \end{pmatrix} \tag{15}$$

with the notation

$$KG^2 = \gamma \tag{16}$$

and the components:

$$\mathbf{c}_{11} = \frac{\gamma^2\cdot(1+K)+\gamma\cdot(5-2K)+6}{4\gamma K}\cdot\exp\left(1-\sqrt{2+\gamma}\right)2t+$$
$$\frac{\gamma^3+(K+2)\cdot\gamma^2+(1+7K)\cdot\gamma+2}{4\gamma K}\cdot\exp\left(1+\sqrt{2+\gamma}\right)2t-$$
$$\frac{(K+1)\cdot\gamma^2+(K-3)\cdot\gamma+6K+4}{2\gamma K}\cdot\exp\left(2t\right),$$

$$\mathbf{c}_{12} = \frac{1}{4}\cdot\left[\frac{\left(-\gamma^2-\gamma-2\right)\cdot\sqrt{2+\gamma}-4\gamma-4}{\gamma(2+\gamma)}\right]\cdot\exp\left(1-\sqrt{2+\gamma}\right)2t+$$
$$\frac{1}{4}\cdot\left[\frac{(3\gamma+6)\cdot\sqrt{2+\gamma}-2\gamma^3-2\gamma^2+3\gamma+2}{\gamma(2+\gamma)}\right]\cdot\exp\left(-1-\gamma\right)t+$$
$$\frac{1}{4}\cdot\left[\frac{\left(-2\gamma^2-3\gamma-2\right)\cdot\sqrt{2+\gamma}+2\gamma^3+6\gamma^2+3\gamma-2}{\gamma(2+\gamma)}\right]\cdot\exp\left(1+\sqrt{2+\gamma}\right)2t,$$

$$\mathbf{c}_{21} = \frac{1}{4}\cdot\frac{\gamma\cdot\left(2+\gamma-\sqrt{2+\gamma}\right)}{2+\gamma}\cdot\exp\left(1-\sqrt{2+\gamma}\right)2t+$$
$$\frac{1}{4}\cdot\frac{\gamma\cdot\left(2+\gamma+\sqrt{2+\gamma}\right)}{2+\gamma}\cdot\exp\left(1+\sqrt{2+\gamma}\right)2t-\frac{1}{4}\cdot2\gamma\cdot\exp\left(-1-\gamma\right)t,$$

$$\mathbf{c}_{22} = \frac{1}{4}\cdot\left(\frac{\gamma^2}{2+\gamma}+1\right)\cdot\exp\left(1-\sqrt{2+\gamma}\right)2t+$$
$$\frac{1}{4}\cdot\left[\frac{\gamma^2}{2+\gamma}+\left(1+\sqrt{2+\gamma}\right)^2\right]\cdot\exp\left(1+\sqrt{2+\gamma}\right)2t+$$
$$\frac{1}{4}\cdot\left[-\frac{2\gamma^2}{2+\gamma}+2\left(1-\sqrt{2+\gamma}\right)\right]\cdot\exp\left(-1-\gamma\right)t.$$

In [3], it was evaluated the Cauchy-Green tensor from the trajectories analysis standpoint. The MapleVI soft was used, for plotting the trajectories in discrete time. In what follows, the deformations for this flow are studied, in order to evaluating the efficiency of mixing. For the moment, due to the complexity of calculus, only the length deformation λ^2 is studied. Thus, applying the relation (7), with the above components, it is found:

$$\lambda^2 = \begin{bmatrix} \frac{\gamma^2+5\gamma+6}{4\gamma} \cdot \exp\left(1-\sqrt{2+\gamma}\right)2t+ \\ \frac{\gamma^3+2\gamma^2+\gamma+2}{4\gamma} \cdot \exp\left(1+\sqrt{2+\gamma}\right)2t- \\ \frac{\gamma^2-3\gamma+4}{2\gamma} \cdot \exp\left(2t\right) \end{bmatrix} \cdot \mathbf{M}_1^2 +$$

$$2\begin{bmatrix} \frac{\left(-2\gamma^2-\gamma-2\right)\cdot\sqrt{2+\gamma}+\gamma^3+2\gamma^2-4\gamma-4}{\gamma(2+\gamma)} \cdot \exp\left(1-\sqrt{2+\gamma}\right)2t+ \\ \frac{\left(-\gamma^2-3\gamma-2\right)\cdot\sqrt{2+\gamma}+3\gamma^3+8\gamma^2+3\gamma-2}{\gamma(2+\gamma)} \cdot \exp\left(1+\sqrt{2+\gamma}\right)2t+ \\ \frac{(3\gamma+6)\cdot\sqrt{2+\gamma}-6\gamma^2+3\gamma+2}{\gamma(2+\gamma)} \cdot \exp\left(-1-\gamma\right)t \end{bmatrix} \cdot \mathbf{M}_1\mathbf{M}_2 +$$

$$\begin{bmatrix} \frac{\gamma^2+\gamma+2}{\gamma+2} \cdot \exp\left(1-\sqrt{2+\gamma}\right)2t+ \\ \frac{\gamma^2+(2+\gamma)\left(3+\gamma-2\sqrt{2+\gamma}\right)}{2+\gamma} \cdot \exp\left(1+\sqrt{2+\gamma}\right)2t+ \\ \frac{-2\gamma^2+(4+2\gamma)\left(1-\sqrt{2+\gamma}\right)}{2+\gamma} \cdot \exp\left(-1-\gamma\right)t \end{bmatrix} \cdot \mathbf{M}_2^2, \qquad (17)$$

where $\sum M_i^2 = 1$.

As it can be seen, the calculus is quite complex, some polynomials are involved for each exponential.

Let us consider two irrational, *random*, values for the length versor, namely:

$$(M_1, M_2) = \left(-\frac{1}{\sqrt{3}}, \frac{\sqrt{2}}{\sqrt{3}}\right),$$

$$(M_1, M_2) = \left(\frac{1}{\sqrt{7}}, -\frac{\sqrt{6}}{\sqrt{7}}\right).$$

For each of these cases, it is studied the behavior of the length deformation, *as function of time*, for some values of γ. If we look at the relation (17), some symmetries of its form are observed. Therefore, taking into account some remarks of [3], the following values of γ are taken into account, for the beginning:

$$\begin{array}{ll} A. & \gamma = -0.75; \\ B. & \gamma = -0.05; \\ C. & \gamma = 0.75. \end{array}$$

Using specific procedures of MapleVI soft, there were found 2x3 = 6 plots, three for each versor case. These are continuous time plots, the time is varying between 0 and 20 units. The figures are numbered by the parameter case.

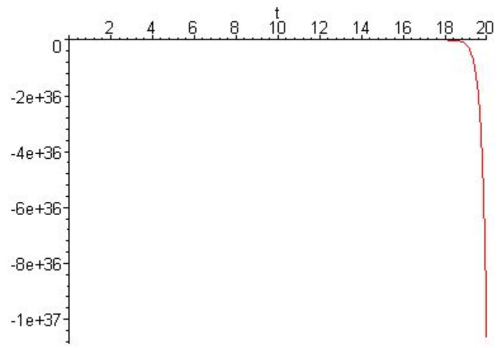

Figure 1: A1

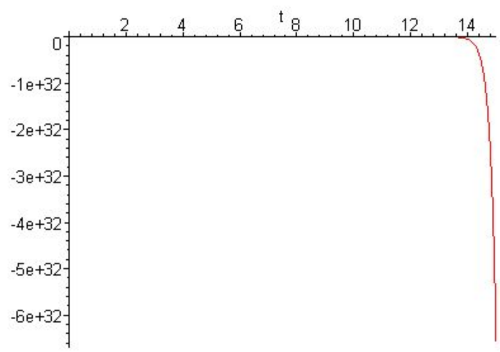

Figure 2: B1

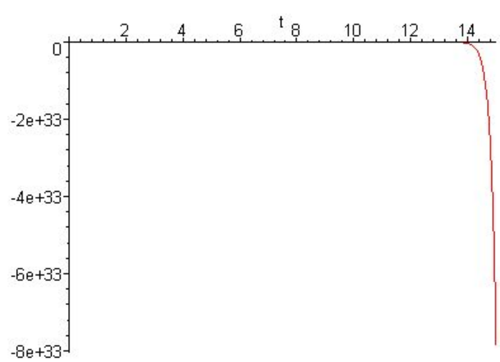

Figure 3: C1

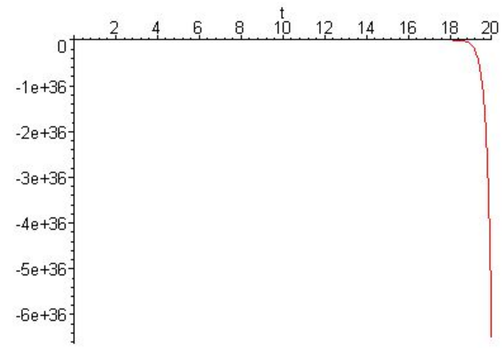

Figure 4: A2

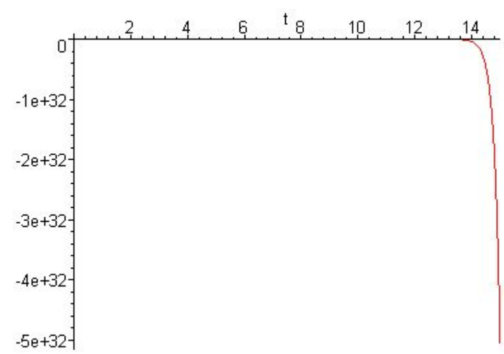

Figure 5: B2

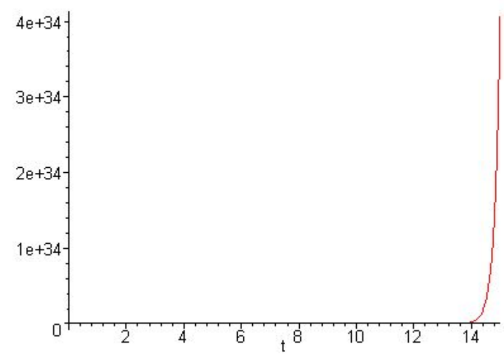

Figure 6: C2

3. Remarks

Looking at the above analysis, some remarks issue:

1. For any case of versor and/ or parameter case, the length deformation λ^2 is *unbounded*. Moreover, for the first versor case, as γ increases, it must be shortened the axes, for avoiding the so-called "*floating point overflow*", that means a breakup of the simulation. This happened also for the second versor case.

2. The length deformation has a *negative behavior*, although only a small time scale was considered. Only in the last case a positive behavior was noted. Thus, comparing to the cases studied in [1], it can be assessed that for a small perturbation of the 2D general model, and on a larger time interval, it becomes also far from equilibrium.

3. It can be assessed, as for 2D periodic case [2], that the *irrational versor values* produce nonlinear phenomena. It is expected that the efficiency of mixing has a more complicated expression.

4. As an immediate aim, more irrational versor values will be taken into account. That will be useful also for studying the efficiency of deformations, in length and also in surface. As perturbing the initial model, the calculus becomes very complex, therefore a parametric approach would be very useful.

5. The analysis of the length deformation for a small perturbation confirms that the flows of the studied type, in 2D and 3D case, *have a chaotic behavior*. This matches the experiments in [6].

References

[1] Ionescu, A., The structural stability of biological oscillators. Analytical contributions, Ph.D. Thesis. Politechnic University of Bucarest, 2002.

[2] Ionescu, A., Costescu, M. "Computational aspects in excitable media. The case of vortex phenomena". Proceedings of ICCCC2006, Oradea, ISSN 1841-1844, pag 280-284

[3] Ionescu, A., Costescu, M., Coman D. "A computational approach of turbulent mixing. The trajectories analysis". Proc. of ICEEMS2007, Brasov (in press)

[4] Ottino, J.M., W.E. Ranz & C.W. Macosko. "A framework for the mechanical mixing of fluids". AlChE J. 27,1981, 565-577

[5] Ottino, J.M. "The kinematics of mixing: stretching, chaos and transport". Cambridge University Press. 1989, ISSN 0521368782

[6] Savulescu, St.N. "Applications of multiple flows in a vortex tube closed at one end". Internal Reports, CCTE, IEA. Bucarest. (1996-1998)

Authors:

Adela Ionescu
Department of Applied Sciences
Faculty of History, Philosophy and Geography
University of Craiova
22 December 1989, No. 8, G5-III-3, 200724 Craiova
e-mail: *adela0404@yahoo.com*

Mihai Costescu
Department of Political Sciences
Faculty of Engineering and Management of Technological Systems
University of Craiova
Str A.I.Cuza, No. 13, Craiova
e-mail: *miracos2003@yahoo.com*