A CLASS OF HOLOMORPHIC FUNCTIONS DEFINED BY INTEGRAL OPERATOR

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Abstract. By using the integral operator $I^m f(z)$, $z \in U$ (Definition 4), we introduce a class of holomorphic functions, denoted by $\mathcal{I}^m(\alpha)$, and we obtain inclusion relations related to this class and some differential subordinations.

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1. Introduction and preliminaries

Denote by U the unit disc of the complex plane:

$$U = \{z \in \mathbb{C} : \ |z| < 1\}.$$

Let $\mathcal{H}[U]$ be the space of holomorphic functions in U.
We let:

$$A_n = \{f \in \mathcal{H}[U], \ f(z) = z + a_{n+1}z^{n+1} + \dots, \ z \in U\}$$

with $A_1 = A$.
We let $\mathcal{H}[a, n]$ denote the class of analytic functions in U of the form

$$f(z) = a + a_n z^n + a_{n+1}z^{n+1} + \dots, \quad z \in U.$$

Let

$$K = \left\{ f \in A : \ \mathrm{Re}\, \frac{zf''(z)}{f'(z)} + 1 > 0, \ z \in U \right\}$$

denote the class of normalized convex functions in U.

If f and g are analytic functions in U, then we say that f is subordinate to g, written $f \prec g$, or $f(z) \prec g(z)$, if there is a function w analytic in U with $w(0) = 0$, $|w(z)| < 1$, for all $z \in U$ such that $f(z) = g[w(z)]$ for $z \in U$. If g is univalent, then $f \prec g$ if $f(0) = g(0)$ and $f(U) \subset g(U)$.

Definition 1 *[1] Let $\psi : \mathbb{C}^3 \times U \to \mathbb{C}$ and let h be univalent in U. If p is analytic in U and satisfies the (second-order) differential subordinations*

$$\psi\left(p\left(z\right), zp'\left(z\right), z^2p''\left(z\right); z\right) \prec h\left(z\right), \tag{1}$$

then p is called a solution of the differential subordination. The univalent function q is called a dominant of the solutions of the differential subordination, if $p \prec q$ for all p satisfying (1). A dominant $\widetilde{q}$ that satisfies $\widetilde{q} \prec p$ for all dominants q of (1) is said to be the best dominant of (1).

Note that the best dominant is unique up to a rotation of U.

We use the following subordination results:

Lemma 2 *(Hallenbeck and Ruscheweyh [1]) Let h be a convex function with $h(0) \equiv a$ and let $\gamma \in \mathbb{C}^*$ be a complex number with* $\operatorname{Re} \gamma \geq 0$. *If $p \in \mathcal{H}[U]$ with $p(0) = a$ and*

$$p(z) + \frac{1}{\gamma} zp'(z) \prec h(z)$$

then

$$p(z) \prec g(z) \prec h(z)$$

where

$$g(z) = \frac{\gamma}{nz^{\frac{\gamma}{n}-1}} \int_0^z h(t)t^{\frac{\gamma}{n}-1}dt.$$

The function g is convex and is the best dominant.

Lemma 3 *(Miller and Mocanu [2]) Let g be a convex function in U and let*

$$h(z) = g(z) + n\alpha zg'(z)$$

where $\alpha > 0$ and n is a positive integer.

If $p(z) = g(0) + p_nz^n + \dots$ is holomorphic in U and

$$p(z) + \alpha zp'(z) \prec h(z),$$

then

$$p(z) \prec g(z)$$

and this result is sharp.

Definition 4 *[5] For $f \in A$ and $m \in \mathbb{N}$ we define the operator $I^m f$ by*

$$I^0 f(z) = f(z)$$

$$I^1 f(z) = If(z) = \int_0^z f(t)\, t^{-1} dt$$

$$I^m f(z) = I\left(I^{m-1} f(z)\right), \ z \in U.$$

Remark 5 *If $f \in \mathcal{H}(U)$ then $I^m f(z) = \sum_{j=1}^{\infty} j^{-m} a_j z^j$.*

Remark 6 *For $m = 1, I^m f$ is the Alexander operator.*

2.Main results

Definition 7 *If $0 \leq \alpha < 1$ and $m \in \mathbb{N}$, let $\mathcal{I}^m(\alpha)$ denote the class of functions $f \in A$ which satisfy the inequality:*

$$\mathrm{Re}\, [I^m f(z)]' > \alpha.$$

Remark 8 *For $m = 0$, we obtain*

$$Re f'(z) > \alpha, \quad z \in U.$$

Theorem 9 *If $0 \leq \alpha < 1$ and $m \in \mathbb{N}$, then we have*

$$\mathcal{I}^m(\alpha) \subset \mathcal{I}^{m+1}(\delta), \tag{2}$$

where

$$\delta = \delta(\alpha) = 2\alpha - 1 + 2(1-\alpha)\ln 2. \tag{3}$$

The result is sharp.

Proof. Assume that $f \in \mathcal{I}^m(\alpha)$. Then we have

$$I^m f(z) = z[I^{m+1} f(z)]', \quad z \in U. \tag{4}$$

and differentiating this equality we obtain

$$[I^m f(z)]' = \left[I^{m+1} f(z)\right]' + z\left[I^{m+1} f(z)\right]'', \quad z \in U. \tag{5}$$

If $p(z) = [I^{m+1}f(z)]'$, then (5) becomes

$$[I^m f(z)]' = p(z) + zp'(z), \quad z \in U.$$

Since $f \in \mathcal{I}^m(\alpha)$, from definition 7 we have

$$\text{Re } [p(z) + zp'(z)] > \alpha,$$

which is equivalent to

$$p(z) + zp'(z) \prec \frac{1 + (2\alpha - 1)z}{1 + z} \equiv h(z).$$

Therefore, from lemma 2 results that

$$p(z) \prec g(z) \prec h(z), \quad z \in U$$

where

$$g(z) = \frac{1}{z}\int_0^z \frac{1 + (2\alpha - 1)t}{1 + t}dt$$
$$= 2\alpha - 1 + 2(1 - \alpha)\frac{\ln(1 + z)}{z}, \quad z \in U.$$

Moreover, the function g is convex and is the best dominant.

From $p(z) \prec q(z),\ z \in U$, results that

$$\text{Re } p(z) > \text{Re } g(1) = \delta(\alpha) = 2\alpha - 1 + 2(1 - \alpha)\ln 2,$$

hence we can prove that $\mathcal{I}^m(\alpha) \subset \mathcal{I}^{m+1}(\delta)$.

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Theorem 10 *Let g be a convex function, $g(0) = 1$ and let h be a function such that*

$$h(z) = g(z) + zg'(z), \quad z \in U.$$

If $f \in A$ verifies the differential subordination

$$[I^m f(z)]' \prec h(z), \quad z \in U \tag{6}$$

then

$$[I^{m+1} f(z)]' \prec g(z), \quad z \in U$$

and this result is sharp.

Proof. By using the properties of the operator $I^m f$ and differentiating we obtain

$$[I^m f(z)]' = \left[I^{m+1} f\left(z\right)\right]' + z\left[I^{m+1} f(z)\right]'', \quad z \in U.$$

If we let

$$p(z) = [I^{m+1} f(z)]',$$

then we obtain

$$[I^m f(z)]' = p(z) + zp'(z)$$

and (6) becomes

$$p(z) + zp'(z) \prec g(z) + zg'(z) \equiv h(z).$$

By using lemma 3, we have

$$p(z) \prec g(z),$$

i.e.

$$\left[I^{m+1} f\left(z\right)\right]' \prec g\left(z\right), \quad z \in U$$

and this result is sharp.

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Theorem 11 *Let g be a convex function, $g(0) = 1$, and*

$$h(z) = g(z) + zg'(z), \quad z \in U.$$

If $f \in A$ and verifies the differential subordination

$$[I^m f(z)]' \prec h(z), \quad z \in U, \tag{7}$$

then

$$\frac{I^m f\left(z\right)}{z} \prec g\left(z\right), \quad z \in U$$

where

$$g\left(z\right) = \frac{1}{z}\int_0^z h\left(t\right) dt, \quad z \in U$$

and this result is sharp.

Proof. If

$$p(z) = \frac{I^m f(z)}{z}, \ z \in U \tag{8}$$

then results that

$$I^m f(z) = zp(z). \tag{9}$$

Differentiating (9), we obtain

$$[I^m f(z)]' = p(z) + zp'(z), \quad z \in U,$$

hence (7) becomes

$$p(z) + zp'(z) \prec h(z) \equiv g(z) + zg'(z).$$

Therefore, from lemma 2 results that

$$p(z) \prec g(z), \ z \in U$$

i.e.

$$\frac{I^m f(z)}{z} \prec g(z), \quad z \in U.$$

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Theorem 12 *Let $h \in \mathcal{H}[U]$, with $h(0) = 1$, $h'(0) \neq 0$, which verifies the inequality*

$$\mathrm{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}, \quad z \in U.$$

If $f \in A$ and verifies the differential subordination

$$[I^m f(z)]' \prec h(z), \quad z \in U, \tag{10}$$

then

$$[I^{m+1} f(z)]' \prec g(z)$$

where

$$g(z) = \frac{1}{z}\int_0^z h(t)dt, \ z \in U.$$

The function g is convex and is the best dominant.

Proof. A simple application of the differential subordinations technique [1, Corollary 2.6g.2, p. 66] shows that the function g is convex. From

$$I^m f(z) = z[I^{m+1} f(z)]', \; z \in U$$

we obtain

$$[I^m f(z)]' = \left[I^{m+1} f\,(z)\right]' + z\left[I^{m+1} f\,(z)\right]''.$$

If we assume

$$p(z) = [I^{m+1} f(z)]'$$

then

$$[I^m f(z)]' = p(z) + zp'(z), \; z \in U,$$

hence (10) becomes

$$p(z) + zp'(z) \prec h(z).$$

Moreover, from lemma 2 results that

$$p(z) \prec g(z) = \frac{1}{z}\int_0^z h(t)dt.$$

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Theorem 13 *Let $h \in \mathcal{H}(U)$, $h(0) = 1$, $h'(0) \neq 0$, which satisfy the inequality*

$$\mathrm{Re}\left[1 + \frac{zh''(z)}{h'(z)}\right] > -\frac{1}{2}, \quad z \in U.$$

If $f \in A$ and verifies the differential subordination

$$[I^m f(z)]' \prec h(z), \quad z \in U, \tag{11}$$

then

$$\frac{I^m f(z)}{z} \prec g(z), \quad z \in U, \; z \neq 0,$$

where

$$g(z) = \frac{1}{z}\int_0^z h(t)tdt, \; z \in U.$$

The function g is convex and is the best dominant.

Proof. A simple application of the differential subordinations technique [1, Corollary 2.6g.2, p. 66] shows that the function g is convex.

If

$$p(z)=\frac{I^m f(z)}{z},\quad z\in U,\ z\neq 0. \tag{12}$$

then differentiating relation (12) we obtain

$$[I^m f(z)]'=p(z)+zp'(z),$$

so (11) becomes

$$p(z)+zp'(z)\prec h(z),\quad z\in U.$$

Therefore, from lemma 2 results that $p(z)\prec g(z)$ where

$$g(z)=\frac{1}{z}\int_0^z h(t)dt,\ z\in U,$$

and g is convex and is the best dominant. ∎

Similar results for differential operator were obtained by G. I. Oros in [4].

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