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FIXED POINT THEORY FOR CIRIC-TYPE-GENERALIZED φ-PROBABILISTIC CONTRACTION IN PROBABILISTIC MENGER SPACES

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Abstract. In this paper, we introduce Ciric-type-generalized φ-probabilistic contraction in probabilistic Menger spaces. We derive some results about existence and uniqueness of a fixed point for this class of self mappings in probabilistic Menger spaces.

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1. Introduction and Preliminaries

Probabilistic metric space (abbreviated, PM space) has been introduced and studied in 1942 by Karl Menger in [10]. The idea of Menger was to use distribution functions instead of nonnegative real numbers as values of the metric. The notion of a PM space corresponds to the situation when we do not know exactly the distance between two points, we know only probabilities of possible values of this distance. In fact the study of such spaces received an impetus with the pioneering works of Schweizer and Sklar [12] and [13]. Recently, the study of fixed point theorems in PM spaces is also a topic of recent interest and forms an active direction of research. Sehgal et al. [14] made the first ever effort in this direction. Since then, several authors have already studied fixed point, common fixed point theorems and recently best proximity point in PM spaces. The contraction principle was probabilistically generalized by Sehgal and Bharucha-Reid in [14]. This probabilistic generalization of the contraction inequality has come to be known as Sehgal's contraction. Their result was proved in the context of probabilistic metric spaces [12]. After that, the fixed point theory in probabilistic metric spaces is developed immensely in different directions. A comprehensive description of which also has been given in the book by Hadžić

and Pap [4]. Some other recent references are noted in [5, 6, 9, 11, 15, 16, 17, 19]. In this paper, we introduce Ciric-type-generalized φ-probabilistic contraction in probabilistic Menger spaces. We derive some results about existence and uniqueness of a fixed point for this class of self mappings in probabilistic Menger spaces.
First we shall recall some well-known definitions and results in the theory of PM spaces which are used later on in this paper. For more details, we refer the reader to [4] and [12].

Throughout this paper, we denote $\mathbb{R}$ the set of all real numbers, and by $\mathbb{R}^+$ the set of all nonnegative real numbers.

Definition 1. *A distribution function is a function $F : [-\infty, \infty] \to [0, 1]$, that is left continuous on $\mathbb{R}$ and nondecreasing moreover, $F(-\infty) = 0$ and $F(\infty) = 1$.*

The set of all the distribution functions is denoted by Δ, and the set of those distribution functions such that $F(0) = 0$ is denoted by Δ^+. The set of all $F \in \Delta^+$ for which $\lim_{t\to\infty} F(t) = 1$ will be denoted by D^+. The space Δ^+ is partially ordered by the usual pointwise ordering of functions, and has a maximal element ϵ_0, defined by

$$\epsilon_0(t) = \begin{cases} 0 & t \leq 0, \\ 1 & t > 0. \end{cases}$$

Definition 2. *A probabilistic metric space (abbreviated, PM space) is an ordered pair (X, F), where X is a nonempty set and $F : X \times X \to \Delta^+$ ($F(x, y)$ is denoted by $F_{x,y}$) satisfies the following conditions:*

(PM1) $F_{x,y} = \epsilon_0$, *iff* $x = y$,

(PM2) $F_{x,y} = F_{y,x}$,

(PM3) *If* $F_{x,y}(t) = 1$ *and* $F_{y,z}(s) = 1$, *then* $F_{x,z}(t + s) = 1$,

for every $x, y, z \in X$ and $t, s \geq 0$.

Definition 3. *A mapping $\Delta : [0, 1] \times [0, 1] \to [0, 1]$ is called a triangular norm (abbreviated, t-norm) if the following conditions are satisfied:*

(i) $\Delta(a, b) = \Delta(b, a)$,

(ii) $\Delta(a, \Delta(b, c)) = \Delta(\Delta(a, b), c)$,

(iii) $\Delta(a, b) \geq \Delta(c, d)$ *whenever* $a \geq c$ *and* $b \geq d$,

(iv) $\Delta(a, 1) = a$,

for every $a, b, c, d \in [0, 1]$.

Two typical examples of continuous t-norm are $\Delta_p(a,b) = ab$ and $\Delta_m(a,b) = \min\{a,b\}$. It is evident that, as regards the pointwise ordering, $\Delta \leq \Delta_m$, for each t-norm Δ.

An arbitrary t-norm can be extended (by (iii)) in a unique way to an n-ary operation. For $(a_1,\cdots,a_n) \in [0,1]^n$ $(n \in \mathbb{N})$, the value $\Delta^n(a_1,\cdots,a_n)$ is defined by $\Delta^1(a_1) = a_1$ and $\Delta^n(a_1,\cdots,a_n) = \Delta(\Delta^{n-1}(a_1,\cdots,a_{n-1}),a_n)$. For each $a \in [0,1]$, the sequence $(\Delta^n(a))$ is defined by $\Delta^n(a) = \Delta^n(a,\cdots,a)$.

Definition 4. *A t-norm Δ is said to be of Hadžić type (abbreviated, H-type) if the sequence of functions $(\Delta^n(a))$ is equicontinuous at $a = 1$, that is*

$$\forall\, \varepsilon \in (0,1), \quad \exists\, \delta \in (0,1): \; a > 1-\delta \Rightarrow \Delta^n(a) > 1-\varepsilon, \quad (n \in \mathbb{N}).$$

The t-norm Δ_m is a trivial example of a t-norm of H-type, but there are t-norms Δ of H-type with $\Delta \neq \Delta_m$, see [4]. It is easy to see that if Δ is of H-type, then Δ satisfies $\sup_{a\in(0,1)} \Delta(a,a) = 1$.

Definition 5. *A probabilistic Menger space is a triplet (X,F,Δ), where (X,F) is PM space and Δ is a t-norm such that for all $x,y,z \in X$ and for all $t,s \geq 0$,*

$$F_{x,z}(t+s) \geq \Delta(F_{x,y}(t), F_{y,z}(s)).$$

Definition 6. *Let (X,F,Δ) be a probabilistic Menger space. An open ball with center x and radius λ $(0 < \lambda < 1)$ in X is the set $U_x(\varepsilon,\lambda) = \{y \in X : F_{x,y}(\varepsilon) > 1-\lambda\}$, for all $\varepsilon > 0$. It is easy to see that $\mathfrak{U} = \{U_x(\varepsilon,\lambda) : \; x \in X, \; \varepsilon > 0, \; \lambda \in (0,1)\}$ determines a Hausdorff topology for X [12, 12.1.2].*

Definition 7. *A sequence (x_n) in a probabilistic Menger space (X,F,Δ) is said to be convergent to a point $x \in X$ if and only if for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists $n_0(\varepsilon,\lambda) \in \mathbb{N}$ such that $F_{x_n,x}(\varepsilon) > 1-\lambda$ for all $n \geq n_0(\varepsilon,\lambda)$, or $\lim_{n\to\infty} F_{x_n,x}(t) = 1$, for all $t > 0$. In this case we say that limit of the sequence (x_n) is x.*

Definition 8. *A sequence (x_n) in a probabilistic Menger space (X,F,Δ) is said to be Cauchy sequence if and only if for every $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists $n_0(\varepsilon,\lambda) \in \mathbb{N}$ such that $F_{x_{n+p},x_n}(\varepsilon) > 1-\lambda$ for all $n \geq n_0(\varepsilon,\lambda)$ and every $p \in \mathbb{N}$, or $\lim_{n\to\infty} F_{x_{n+p},x_n}(t) = 1$, for all $t > 0$ and $p \in \mathbb{N}$.*

Also, a probabilistic Menger space (X,F,Δ) is said to be complete if and only if every Cauchy sequence in X is convergent.

The concept of Cauchy sequence is inspired from that of G-Cauchy sequence (it belongs to Grabiec [3]).

Proof. The limit of a convergent sequence in a probabilistic Menger space (X, F, Δ) is unique.

Proof. It is obvious.

Proof. [1, 2.5.3] If (X, F, Δ) is a probabilistic Menger space and Δ is continuous, then probabilistic distance function F is a low semi continuous function of points, i.e. for every fixed point $t > 0$, if $x_n \to x$ and $y_n \to y$, then

$$\liminf_{n\to\infty} F_{x_n,y_n}(t) = F_{x,y}(t).$$

Proof. [21, Lemma 2. 2] Let $n \in \mathbb{N}$, $F \in \Delta^+$ and $g_1, g_2, \ldots g_n : \mathbb{R} \to [0, 1]$. If $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a mapping such that $\varphi(t) < t$, $\lim_{n\to\infty} \varphi^n(t) = 0$ and

$$F(\varphi(t)) \geq \min\{g_1(t), g_2(t), \ldots g_n(t), F(t)\}, \qquad \forall t > 0.$$

Then $F(\varphi(t)) \geq \min\{g_1(t), g_2(t), \ldots g_n(t)\}$ for all $t > 0$.

The following lemma has been proved by Jachymski in [8], for mappings $g_n : (0, \infty) \to (0, \infty)$, but it is also valid for mappings $g_n : [0, \infty) \to [0, \infty)$.

Lemma 1. *[8] Let $n \in \mathbb{N}$, $g_n : [0, \infty) \to [0, \infty)$ and $F_n, F : \mathbb{R} \to [0, 1]$. Assume that $\sup\{F(t) : t > 0\} = 1$ and for any $t > 0$, $\lim_{n\to\infty} g_n(t) = 0$ and $F_n(g_n(t)) \geq F(t)$. If each F_n is nondecreasing, then $\lim_{n\to\infty} F_n(t) = 1$ for any $t > 0$.*

Lemma 2. *Let (X, F, Δ) be a probabilistic Menger space and $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ be a mapping such that $\varphi(0) = 0$, $\varphi(t) < t$ and $\lim_{n\to\infty} \varphi^n(t) = 0$. If $x, y \in X$ and $F_{x,y}(\varphi(t)) \geq F_{x,y}(t)$ for all $t > 0$. Then $x = y$.*

Proof. By using the above lemma, the result follows.

The probabilistic version of the classical Banach contraction principle, was first studied in 1972 by Sehgal and Bharucha-Reid [14].

Theorem 3. *[14] Let (X, F, Δ_m) be a complete probabilistic Menger space. If T is a contraction mapping of X into itself, that is there exists a constant $0 < c < 1$ such that*

$$F_{Tx,Ty}(ct) \geq F_{x,y}(t), \qquad \forall t > 0, x, y \in X.$$

Then there is a unique $x^* \in X$ such that $Tx^* = x^*$. Moreover, $(T^n x_0)$ converges to x^* for each $x_0 \in X$.

Definition 9. *[4, p. 98] Let (X, F) be a PM space and $T : X \to X$ be a mapping. For every $x_0 \in X$, the orbit of the mapping T at x_0 is $O(x_0, T) = \{T^n x_0 : n \in \mathbb{N} \cup \{0\}\}$. Let $D_{O(x_0,T)} : \mathbb{R} \to [0, 1]$ be a diameter of $O(x_0, T)$, i.e, $D_{O(x_0,T)}(t) = \sup_{s<t} \inf_{x,y \in O(x_0,T)} F_{x,y}(s)$. If $\sup_{t\in\mathbb{R}} D_{O(x_0,T)}(t) = 1$, then the orbit $O(x_0, T)$ is a probabilistic bounded subset of X. Hence $O(x_0, T)$ is a probabilistic bounded set if and only if $D_{O(x_0,T)} \in D^+$. Also, X is said to be T-orbitally complete if for all $x \in X$, $O(x, T)$ is complete.*

In recent years, a number of generalizations of the Banach contraction principle have appeared. Of all these, the following generalization of Ciric [2] stands at the top.

Theorem 4. *[2] Let (X, F, Δ_m) be a complete probabilistic Menger space. If $T : X \to X$ is generalized contraction mapping on X, that is there exists a constant $0 < c < 1$ such that for every $x, y \in X$*

$$F_{Tx,Ty}(ct) \geq \min\{F_{x,y}(t), F_{x,Tx}(t), F_{y,Ty}(t), F_{x,Ty}(t), F_{Tx,y}(t)\},$$

for all $t > 0$, and X is T-orbitally complete. Then there is a unique $x^* \in X$ such that $Tx^* = x^*$. Moreover, $(T^n x_0)$ converges to x^* for each $x_0 \in X$.

Theorem 5. *[8] Let (X, F, Δ) be a complete probabilistic Menger space under a t-norm Δ of H-type. If $T : X \to X$ is a generalized φ-probabilistic contraction, that is,*

$$F_{Tx,Ty}(\varphi(t)) \geq F_{x,y}(t), \qquad \forall t > 0,\ \forall x, y \in X. \tag{1}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a mapping such that, for any $t > 0$, $0 < \varphi(t) < t$ and $\lim\limits_{n\to\infty} \varphi^n(t) = 0$. Then, there is a unique $x^* \in X$ such that $Tx^* = x^*$. Moreover, $(T^n x_0)$ converges to x^* for each $x_0 \in X$.

Definition 10. *Let (X, F, Δ) be a probabilistic Menger space and $T : X \to X$. We say that T is Ciric-type-generalized φ-probabilistic contraction if for every $x, y \in X$ and $t > 0$,*

$$F_{Tx,Ty}(\varphi(t)) \geq \min\{F_{x,y}(t), F_{x,Tx}(t), F_{y,Ty}(t), F_{x,Ty}(t), F_{Tx,y}(t)\}, \tag{2}$$

where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a mapping.

The following example due to Ume [20] shows that a Ciric-type-generalized φ-probabilistic contraction need not be a generalized φ-probabilistic contraction.

Example 1. *Let $X = [0, \infty)$, $T : X \to X$ be defined by $Tx = x + 1$, and let $\varphi : [0, \infty) \to [0, \infty)$ be defined by*

$$\varphi(t) = \begin{cases} \frac{t}{1+t}, & 0 \leq t \leq 1, \\ t - 1, & 1 < t. \end{cases}$$

For each $x, y \in X$*, let* $F_{x,y}(t) = \epsilon_0(t- \mid x-y \mid)$ *for all* $t \in \mathbb{R}$*. Since* $\max\{\mid x-y-1 \mid, \mid y-x-1 \mid\} = \mid x-y \mid +1$ *for all* $x, y \in X$*, then* $F_{Tx,Ty}(\varphi(t)) \geq \min\{F_{x,Ty}(t), F_{Tx,y}(t)\}$*. Thus,*

$$F_{Tx,Ty}(\varphi(t)) \geq \min\{F_{x,y}(t), F_{x,Tx}(t), F_{y,Ty}(t), F_{x,Ty}(t), F_{Tx,y}(t)\},$$

which satisfies (2)*. If* $t = 2, x = 0$ *and* $y = \frac{3}{2}$*, then* $F_{T0,T\frac{3}{2}}(\varphi(2)) = 0$ *and* $F_{0,\frac{3}{2}}(2) = 1$*. Thus,* $F_{T0,T\frac{3}{2}}(\varphi(2)) < F_{0,\frac{3}{2}}(2)$*, which does not satisfy* (1)*.*

As the following example due to Ume [20] shows, there exists T that does not satisfy (2) with $\varphi(t) = kt$, $0 < k < 1$.

Example 2. *Let* $X = [0, \infty)$*,* $T : X \to X$ *be defined by* $Tx = 2x$*, and let* $\varphi : [0, \infty) \to [0, \infty)$ *be defined by* $\varphi(t) = kt$*,* $0 < k < 1$*. For each* $x, y \in X$*, let* $F_{x,y}(t) = \epsilon_0(t- \mid x-y \mid)$ *for all* $t \in \mathbb{R}$*. If* $x = 0, y = 1$ *and* $t = \frac{2}{k} > 0$*, then for simple calculations,* $F_{T0,T1}(\varphi(\frac{2}{k})) = 0$ *and*

$$\min\left\{F_{0,1}(\frac{2}{k}), F_{0,T0}(\frac{2}{k}), F_{1,T1}(\frac{2}{k}), F_{0,T1}(\frac{2}{k}), F_{T0,1}(\frac{2}{k})\right\} = 1.$$

Therefore, for $x = 0, y = 1$ *and* $t = \frac{2}{k} > 0$*, the mapping* T *does not satisfy* (2)*. Thus, we showed that there exists* T *that does not satisfy* (2) *with* $\varphi(t) = kt$*,* $0 < k < 1$*.*

2. Main results

We first state the following theorem, which appeared in [20], and then we prove that theorem under general conditions.

Theorem 6. *[20] Let* (X, F, Δ_m) *be a probabilistic Menger space and let* T *be a Ciric-type-generalized* φ*-probabilistic contraction mapping such that*

$$F_{x,Tx}((I-\varphi)(t)) \leq \inf\{F_{T^k x, T^l x}(t) : k, l \in \mathbb{Z}^+\}, \quad \forall t > 0,$$

for some $x \in X$ *where* $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ *is a mapping such that*

(i) $\varphi(t) < t$ *for all* $t > 0$*,* $\lim_{t\to\infty}(I-\varphi)(t) = \infty$*, where* $I : \mathbb{R}^+ \to \mathbb{R}^+$ *is identity mapping,*

(ii) φ *and* $I - \varphi$ *are strictly increasing and onto mappings,*

(iii) $\lim_{n\to\infty} \varphi^{-n}(t) = \infty$ *for each* $t > 0$*, where* φ^{-n} *is n-time repeated composition of* φ^{-1} *with itself.*

If (X, F, Δ_m) is T orbitally complete, then T has a unique fixed point in X.

Theorem 7. *Let (X, F, Δ) be a complete probabilistic Menger space under a continuous t-norm Δ and let $T : X \to X$ be a Ciric-type-generalized φ-probabilistic contraction mapping where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a bijective mapping such that $0 < \varphi(t) < t$ and $\lim_{n\to\infty} \varphi^n(t) = 0$ for each $t > 0$. If there exists $x_0 \in X$ with the bounded orbit, then there is a unique $x^* \in X$ such that $Tx^* = x^*$. Moreover, $(T^n x_0)$ converges to x^*.*

Proof. Let $u_n = T^n x_0$, if there exists $n \in \mathbb{N}$, such that $u_{n+1} = u_n$, then there is a $x^* \in X$ such that $Tx^* = x^*$ and $(T^n x_0)$ converges to x^*. So we can assume that $u_{n+1} \neq u_n$ for all $n \in \mathbb{N}$.

Now by the condition (2), we have

$$\begin{aligned} F_{u_n,u_{n+1}}(\varphi(t)) \geq \min\{&F_{u_{n-1},u_n}(t), F_{u_{n-1},u_n}(t), F_{u_n,u_{n+1}}(t), \\ &F_{u_{n-1},u_{n+1}}(t), F_{u_n,u_n}(t)\}, \end{aligned}$$

for all $t \geq 0$, so

$$F_{u_n,u_{n+1}}(\varphi(t)) \geq \min\{F_{u_{n-1},u_n}(t), F_{u_n,u_{n+1}}(t), F_{u_{n-1},u_{n+1}}(t)\}, \quad (\forall t \geq 0). \tag{3}$$

By Proposition 1 we have

$$F_{u_n,u_{n+1}}(\varphi(t)) \geq \min\{F_{u_{n-1},u_n}(t), F_{u_{n-1},u_{n+1}}(t)\}, \quad (\forall t \geq 0). \tag{4}$$

In the following we show by induction that for each $n \in \mathbb{N}$ and for each $t \geq 0$, there exists $1 \leq m \leq n+1$ such that

$$F_{u_n,u_{n+1}}(\varphi^n(t)) \geq F_{u_0,u_m}(t). \tag{5}$$

If $n = 1$, then by (4), we have

$$\begin{aligned} F_{u_1,u_2}(\varphi(t)) &\geq \min\{F_{u_0,u_1}(t), F_{u_0,u_2}(t)\} \\ &= F_{u_0,u_m}(t), \end{aligned}$$

for some $1 \leq m \leq 2$ and for all $t \geq 0$. Thus (5) holds for $n = 1$. Assume towards a contradiction that (5) is not true and take $n_0 > 1$, be the least natural number such that (5) does not hold. So there exists $t_0 > 0$, such that for all $1 \leq m \leq n_0 + 1$, we have

$$F_{u_{n_0},u_{n_0+1}}(t_0) < F_{u_0,u_m}(\varphi^{-n_0}(t_0)). \tag{6}$$

If $\min\{F_{u_{n_0-1},u_{n_0}}(\varphi^{-1}(t_0)), F_{u_{n_0-1},u_{n_0+1}}(\varphi^{-1}(t_0))\} = F_{u_{n_0-1},u_{n_0}}(\varphi^{-1}(t_0))$, then by the hypothesis we have

$$F_{u_{n_0},u_{n_0+1}}(t_0) \geq F_{u_{n_0-1},u_{n_0}}(\varphi^{-1}(t_0)) \geq F_{u_0,u_m}(\varphi^{-n_0}(t_0)),$$

for some $1 \leq m \leq n_0$, a contradiction. Thus

$$\min\{F_{u_{n_0-1},u_{n_0}}(\varphi^{-1}(t_0)), F_{u_{n_0-1},u_{n_0+1}}(\varphi^{-1}(t_0))\} = F_{u_{n_0-1},u_{n_0+1}}(\varphi^{-1}(t_0)).$$

Also form (4), we have

$$F_{u_{n_0},u_{n_0+1}}(t_0) \geq F_{u_{n_0-1},u_{n_0+1}}(\varphi^{-1}(t_0)). \tag{7}$$

By the condition (2), we get

$$\begin{aligned} F_{u_{n_0-1},u_{n_0+1}}(\varphi^{-1}(t)) \geq \min\{&F_{u_{n_0-2},u_{n_0}}(\varphi^{-2}(t)), F_{u_{n_0-2},u_{n_0-1}}(\varphi^{-2}(t)), \\ &F_{u_{n_0},u_{n_0+1}}(\varphi^{-2}(t)), F_{u_{n_0-2},u_{n_0+1}}(\varphi^{-2}(t)), \\ &F_{u_{n_0},u_{n_0-1}}(\varphi^{-2}(t))\}, \end{aligned} \tag{8}$$

for all $t \geq 0$. If

$$\begin{aligned} \min\{&F_{u_{n_0-2},u_{n_0}}(\varphi^{-2}(t_0)), F_{u_{n_0-2},u_{n_0-1}}(\varphi^{-2}(t_0)), F_{u_{n_0-2},u_{n_0+1}}(\varphi^{-2}(t_0)), \\ &F_{u_{n_0},u_{n_0+1}}(\varphi^{-2}(t_0)), F_{u_{n_0},u_{n_0-1}}(\varphi^{-2}(t_0))\} \\ =&F_{u_{n_0},u_{n_0-1}}(\varphi^{-2}(t_0)), \end{aligned}$$

then from (7) and the above, we have

$$\begin{aligned} F_{u_{n_0},u_{n_0+1}}(t_0) \geq& F_{u_{n_0-1},u_{n_0+1}}(\varphi^{-1}(t_0)) \\ \geq& F_{u_{n_0},u_{n_0-1}}(\varphi^{-2}(t_0)) = F_{u_{n_0-1},u_{n_0}}(\varphi^{-2}(t_0)) \\ \geq& F_{u_{n_0},u_m}(\varphi^{-(n_0+1)}(t_0)) \\ \geq& F_{u_{n_0},u_m}(\varphi^{-n_0}(t_0)), \end{aligned}$$

for some $1 \leq m \leq n_0$, a contradiction.

If

$$\begin{aligned} \min\{&F_{u_{n_0-2},u_{n_0}}(\varphi^{-2}(t_0)), F_{u_{n_0-2},u_{n_0-1}}(\varphi^{-2}(t_0)), F_{u_{n_0},u_{n_0+1}}(\varphi^{-2}(t_0)), \\ &F_{u_{n_0-2},u_{n_0+1}}(\varphi^{-2}(t_0))\} = F_{u_{n_0},u_{n_0+1}}(\varphi^{-2}(t_0)), \end{aligned}$$

then form (7), (8) and the above, we have

$$F_{u_{n_0},u_{n_0+1}}(t_0) \geq F_{u_{n_0},u_{n_0+1}}(\varphi^{-2}(t_0)),$$

since $\varphi^{-2}(t_0) > t_0$, then $F_{u_{n_0},u_{n_0+1}}(t_0) = F_{u_{n_0},u_{n_0+1}}(\varphi^{-2}(t_0))$. By (4),

$$F_{u_{n_0},u_{n_0+1}}(\varphi^{-2}(t_0)) \geq \min\{F_{u_{n_0-1},u_{n_0}}(\varphi^{-3}(t_0)), F_{u_{n_0-1},u_{n_0+1}}(\varphi^{-3}(t_0))\}.$$

If $\min\{F_{u_{n_0-1},u_{n_0}}(\varphi^{-3}(t_0)), F_{u_{n_0-1},u_{n_0+1}}(\varphi^{-3}(t_0))\} = F_{u_{n_0-1},u_{n_0}}(\varphi^{-3}(t_0))$, then by the hypothesis we have

$$F_{u_{n_0},u_{n_0+1}}(t_0) \geq F_{u_{n_0-1},u_{n_0}}(\varphi^{-3}(t_0)) \geq F_{u_0,u_m}(\varphi^{-(n_0+2)}(t_0)) \geq F_{u_0,u_m}(\varphi^{-n_0}(t_0)),$$

for some $1 \leq m \leq n_0$, a contradiction. Thus

$$\min\{F_{u_{n_0-1},u_{n_0}}(\varphi^{-3}(t_0)), F_{u_{n_0-1},u_{n_0+1}}(\varphi^{-3}(t_0))\} = F_{u_{n_0-1},u_{n_0+1}}(\varphi^{-3}(t_0)).$$

Also form (4), we have

$$F_{u_{n_0},u_{n_0+1}}(t_0) = F_{u_{n_0},u_{n_0+1}}(\varphi^{-2}(t_0)) \geq F_{u_{n_0-1},u_{n_0+1}}(\varphi^{-3}(t_0)). \tag{9}$$

By the condition (2), we get

$$\begin{aligned} F_{u_{n_0-1},u_{n_0+1}}(\varphi^{-3}(t_0)) \geq \min\{&F_{u_{n_0-2},u_{n_0}}(\varphi^{-4}(t_0)), F_{u_{n_0-2},u_{n_0-1}}(\varphi^{-4}(t_0)),\\ &F_{u_{n_0},u_{n_0+1}}(\varphi^{-4}(t_0)), F_{u_{n_0-2},u_{n_0+1}}(\varphi^{-4}(t_0)),\\ &F_{u_{n_0},u_{n_0-1}}(\varphi^{-4}(t_0))\}. \end{aligned}$$

If

$$\begin{aligned} \min\{&F_{u_{n_0-2},u_{n_0}}(\varphi^{-4}(t_0)), F_{u_{n_0-2},u_{n_0-1}}(\varphi^{-4}(t_0)), F_{u_{n_0},u_{n_0+1}}(\varphi^{-4}(t_0)),\\ &F_{u_{n_0-2},u_{n_0+1}}(\varphi^{-4}(t_0)), F_{u_{n_0},u_{n_0-1}}(\varphi^{-4}(t_0))\} = F_{u_{n_0},u_{n_0-1}}(\varphi^{-4}(t_0)), \end{aligned}$$

then from (9) and the above, we obtain

$$\begin{aligned} F_{u_{n_0},u_{n_0+1}}(t_0) \geq &F_{u_{n_0-1},u_{n_0+1}}(\varphi^{-3}(t_0)) \geq F_{u_{n_0-1},u_{n_0}}(\varphi^{-4}(t_0))\\ \geq &F_{u_0,u_m}(\varphi^{-(n_0+3)}(t_0)) \geq F_{u_0,u_m}(\varphi^{-n_0}(t_0)), \end{aligned}$$

for some $1 \leq m \leq n_0$, a contradiction. If

$$\begin{aligned} \min\{&F_{u_{n_0-2},u_{n_0}}(\varphi^{-4}(t_0)), F_{u_{n_0-2},u_{n_0-1}}(\varphi^{-4}(t_0)), F_{u_{n_0},u_{n_0+1}}(\varphi^{-4}(t_0)),\\ &F_{u_{n_0-2},u_{n_0+1}}(\varphi^{-4}(t_0)), F_{u_{n_0},u_{n_0-1}}(\varphi^{-4}(t_0))\} = F_{u_{n_0},u_{n_0+1}}(\varphi^{-4}(t_0)), \end{aligned}$$

then from (9) and the above, we have

$$F_{u_{n_0},u_{n_0+1}}(t_0) = F_{u_{n_0},u_{n_0+1}}(\varphi^{-4}(t_0)).$$

Again by (4), we have

$$F_{u_{n_0},u_{n_0+1}}(\varphi^{-4}(t_0)) \geq \min\{F_{u_{n_0-1},u_{n_0}}(\varphi^{-5}(t_0)), F_{u_{n_0-1},u_{n_0+1}}(\varphi^{-5}(t_0))\}.$$

Therefore by continuing this process, we see that if

$$\begin{aligned}\min\{&F_{u_{n_0-2},u_{n_0}}(\varphi^{-k}(t_0)), F_{u_{n_0-2},u_{n_0-1}}(\varphi^{-k}(t_0)), F_{u_{n_0},u_{n_0+1}}(\varphi^{-k}(t_0)),\\ &F_{u_{n_0-2},u_{n_0+1}}(\varphi^{-k}(t_0)), F_{u_{n_0},u_{n_0-1}}(\varphi^{-k}(t_0))\} = F_{u_{n_0},u_{n_0-1}}(\varphi^{-k}(t_0)),\end{aligned}$$

for some $k \geq 2$, then

$$F_{u_{n_0},u_{n_0+1}}(t_0) \geq F_{u_0,u_m}(\varphi^{-(n_0+k-1)}(t_0)) \geq F_{u_0,u_m}(\varphi^{-n_0}(t_0)),$$

for some $1 \leq m \leq n_0$, a contradiction. If

$$\begin{aligned}\min\{&F_{u_{n_0-2},u_{n_0}}(\varphi^{-k}(t_0)), F_{u_{n_0-2},u_{n_0-1}}(\varphi^{-k}(t_0)), F_{u_{n_0},u_{n_0+1}}(\varphi^{-k}(t_0)),\\ &F_{u_{n_0-2},u_{n_0+1}}(\varphi^{-k}(t_0)), F_{u_{n_0},u_{n_0-1}}(\varphi^{-k}(t_0))\} = F_{u_{n_0},u_{n_0+1}}(\varphi^{-k}(t_0)),\end{aligned}$$

for all $k \geq 2$, then $F_{u_{n_0},u_{n_0+1}}(t_0) = F_{u_{n_0},u_{n_0+1}}(\varphi^{-k}(t_0))$. Now letting $k \to \infty$, then $F_{u_{n_0},u_{n_0+1}}(t_0) = 1$, which is contradiction with (6). Otherwise, if there exists $k \geq 2$, since $t < \varphi^{-1}(t) < \varphi^{-2}(t) < \cdots$, then we have

$$\begin{aligned}\min\{&F_{u_{n_0-2},u_{n_0}}(\varphi^{-k}(t_0)), F_{u_{n_0-2},u_{n_0-1}}(\varphi^{-k}(t_0)), F_{u_{n_0},u_{n_0+1}}(\varphi^{-k}(t_0)),\\ &F_{u_{n_0-2},u_{n_0+1}}(\varphi^{-k}(t_0)), F_{u_{n_0},u_{n_0-1}}(\varphi^{-k}(t_0))\} \geq\\ \min\{&F_{u_{n_0-2},u_{n_0}}(\varphi^{-2}(t_0)), F_{u_{n_0-2},u_{n_0-1}}(\varphi^{-2}(t_0)), F_{u_{n_0-2},u_{n_0+1}}(\varphi^{-2}(t_0))\}.\end{aligned}$$

Therefore

$$\begin{aligned}F_{u_{n_0-1},u_{n_0+1}}(\varphi^{-1}(t_0)) \geq \min\{&F_{u_{n_0-2},u_{n_0}}(\varphi^{-2}(t_0)), F_{u_{n_0-2},u_{n_0-1}}(\varphi^{-2}(t_0)),\\ &F_{u_{n_0-2},u_{n_0+1}}(\varphi^{-2}(t_0))\}.\end{aligned}$$

From (7) and the above, we get

$$\begin{aligned}F_{u_{n_0},u_{n_0+1}}(t_0) \geq &F_{u_{n_0-1},u_{n_0+1}}(\varphi^{-1}(t_0))\\ \geq &\min\{F_{u_{n_0-2},u_{n_0}}(\varphi^{-2}(t_0)), F_{u_{n_0-2},u_{n_0-1}}(\varphi^{-2}(t_0)),\\ &F_{u_{n_0-2},u_{n_0+1}}(\varphi^{-2}(t_0))\}\\ = &F_{u_{n_0-2},u_m}(\varphi^{-2}(t_0)),\end{aligned} \tag{10}$$

for some $1 \leq m \leq n_0 + 1$. Therefore by continuing this process, we see that for each $1 \leq k \leq n_0$, there exists $1 \leq m \leq n_0 + 1$ such that

$$F_{u_{n_0},u_{n_0+1}}(t_0) \geq F_{u_{n_0-k},u_m}(\varphi^{-k}(t_0)). \tag{11}$$

If $k = n_0$ in (11), then this is a contradiction by (6). So (5) holds for all $n \in \mathbb{N}$. Then from (5) we get

$$F_{u_n,u_{n+1}}(t) \geq F_{u_n,u_{n+1}}(\varphi^n(t)) \geq F_{u_0,u_m}(t) \geq D_{O(x_0,T)}(t).$$

Let $\varepsilon > 0$ and $\lambda \in (0,1)$ be given, since $D_{O(x_0,T)}(t) \to 1$ as $t \to \infty$, then there exists $t_1 > 0$ such that

$$D_{O(x_0,T)}(t_1) > 1 - \lambda.$$

Since $\varphi^n(t_1) \to 0$ as $n \to \infty$, then there is $N \in \mathbb{N}$ such that $\varphi^n(t_1) < \varepsilon$ whenever $n \geq N$. So

$$\begin{aligned} F_{u_n,u_{n+1}}(\varepsilon) &\geq F_{u_n,u_{n+1}}(\varphi^n(t_1)) \\ &\geq D_{O(x_0,T)}(t_1) \\ &> 1 - \lambda. \end{aligned}$$

Thus we proved that for each $\varepsilon > 0$ and for each $\lambda \in (0,1)$, there exists a positive integer N such that

$$F_{u_n,u_{n+1}}(\varepsilon) > 1 - \lambda, \qquad \forall n \geq N.$$

This means that $\lim\limits_{n\to\infty} F_{u_n,u_{n+1}}(t) = 1$ for all $t > 0$. On the other hand

$$F_{u_n,u_{n+p}}(t) \geq \Delta\left(F_{u_n,u_{n+1}}(\frac{t}{p}), F_{u_{n+1},u_{n+2}}(\frac{t}{p}), \ldots, F_{u_{n+p-1},u_{n+p}}(\frac{t}{p})\right), \quad (\forall p \geq 1),$$

now taking the limits as $n \to \infty$, by the hypothesis we get

$$\lim_{n\to\infty} F_{u_n,u_{n+p}}(t) = 1.$$

Hence (u_n) is a Cauchy sequence and by the hypothesis there exists an element $x^* \in X$ such that $\lim\limits_{n\to\infty} u_n = x^*$. Again by (2) we have

$$F_{Tu_n,Tx^*}(\varphi(t)) \geq \min\{F_{u_n,x^*}(t), F_{u_n,Tu_n}(t), F_{x^*,Tx^*}(t), F_{u_n,Tx^*}(t), F_{x^*,Tu_n}(t)\}.$$

Since $\lim\limits_{n\to\infty} Tu_n = \lim\limits_{n\to\infty} u_{n+1} = x^*$, then by Proposition 1 we get

$$F_{x^*,Tx^*}(\varphi(t)) \geq F_{x^*,Tx^*}(t),$$

for all $t \geq 0$ now by Lemma 2, $Tx^* = x^*$. Let $y^* \in X$ such that $Ty^* = y^*$ then from (2) we have

$$\begin{aligned} F_{Tx^*,Ty^*}(\varphi(t)) \geq& \min\{F_{x^*,y^*}(t), F_{Tx^*,x^*}(t), F_{Ty^*,y^*}(t), F_{Tx^*,y^*}(t), F_{x^*,Ty^*}(t)\} \\ =& F_{x^*,y^*}(t), \end{aligned}$$

then $F_{x^*,y^*}(\varphi(t)) \geq F_{x^*,y^*}(t)$, now by Lemma 2, $x^* = y^*$, so the desired result is obtained.

As the following example due to Sherwood [18] shows, the condition-hypothesis that there exists $x_0 \in X$ such that $O(x_0, T)$ is bounded of above theorem, it is necessary condition.

Example 3. *Let G be the distribution function defined by*

$$G(t) = \begin{cases} 0, & t \leq 4, \\ 1 - \frac{1}{n}, & 2^n < t \leq 2^{n+1}, \qquad (n > 1). \end{cases}$$

Consider the set $X = \{1, 2, \ldots, n, \ldots\}$ and define F on $X \times X$ as follows

$$F_{n,n+m}(t) = \begin{cases} 0, & t \leq 0, \\ \Delta_L^m \left(G(2^n t), G(2^{n+1} t), \ldots, G(2^{n+m} t) \right), & t > 0. \end{cases}$$

Where $\Delta_L(a,b) = \max\{a+b-1, 0\}$ for all $a, b \in [0,1]$, then (X, F, Δ_L) is a complete probabilistic Menger space and the mapping $Tx = x+1$ is generalized φ-probabilistic contraction with $\varphi(t) = \frac{t}{2}$. But T is fixed point free mapping. Since there does not exist x in X, such that $O(x, T)$ is bounded.

Example 4. *Consider $X = [-1, 1]$ and define $F_{x,y}(t) = \epsilon_0(t - d(x,y))$ for all $x, y \in X$, where d is Euclidean metric. Then (X, F, Δ_m) is a complete probabilistic Menger space. Define self mapping T on X as follows:*

$$Tx = \begin{cases} 0\ ; & -1 \leq x < 0, \\ \frac{x}{16(1+x)}\ ; & 0 \leq x < \frac{4}{5} \text{ or } \frac{7}{8} < x \leq 1, \qquad (\forall\ x \in [0,1]). \\ \frac{x}{16}\ ; & \frac{4}{5} \leq x \leq \frac{7}{8}, \end{cases}$$

To verify T is generalized φ-probabilistic contraction with $\varphi(t) = \frac{1}{8}t$, we need to consider several possible cases.

Case 1. Let $x, y \in [-1, 0)$. Then

$$d(Tx, Ty) = \mid Tx - Ty \mid = 0 \leq \frac{1}{8} \mid x - y \mid = \frac{1}{8} d(x, y).$$

Case 2. Let $x \in [-1, 0)$ and $y \in [0, \frac{4}{5}) \cup (\frac{7}{8}, 1]$. Then

$$d(Tx, Ty) = \mid Tx - Ty \mid = \frac{y}{16(1+y)} \leq \frac{1}{8} \mid y - 0 \mid = \frac{1}{8} d(y, Tx).$$

Case 3. Let $x \in [-1, 0)$ and $y \in [\frac{4}{5}, \frac{7}{8}]$. Then

$$d(Tx, Ty) = \mid Tx - Ty \mid = \frac{y}{16} \leq \frac{1}{8} \mid y - 0 \mid = \frac{1}{8} d(y, Tx).$$

Case 4. Let $x, y \in [0, \frac{4}{5}) \cup (\frac{7}{8}, 1]$. *Then*

$$d(Tx, Ty) = \mid Tx - Ty \mid = \mid \frac{x}{16(1+x)} - \frac{y}{16(1+y)} \mid \leq \frac{1}{8} \mid x - y \mid = \frac{1}{8} d(x, y).$$

Case 5. Let $x \in [0, \frac{4}{5}) \cup (\frac{7}{8}, 1]$ *and* $y \in [\frac{4}{5}, \frac{7}{8}]$. *Then*

$$d(Tx, Ty) = \mid Tx - Ty \mid = \mid \frac{x}{16(1+x)} - \frac{y}{16} \mid \leq \frac{1}{16}(\frac{x}{1+x} + y) \leq \frac{1}{16}(\frac{1}{2} + \frac{7}{8}) \leq \frac{11}{128},$$

and

$$\frac{123}{160} = \frac{4}{5} - \frac{1}{16}\frac{1}{2} \leq y - \frac{x}{16(1+x)} \leq \mid y - \frac{x}{16(1+x)} \mid = d(y, Tx).$$

Thus

$$d(Tx, Ty) \leq \frac{11}{128} \leq \frac{123}{1280} = \frac{1}{8} \times \frac{123}{160} \leq \frac{1}{8} d(y, Tx).$$

Case 6. Let $x, y \in [\frac{4}{5}, \frac{7}{8}]$. *Then*

$$d(Tx, Ty) = \mid Tx - Ty \mid = \mid \frac{x}{16} - \frac{y}{16} \mid \leq \frac{1}{8} \mid x - y \mid = \frac{1}{8} d(x, y).$$

Hence, we obtain

$$d(Tx, Ty) \leq \frac{1}{8} \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}, \quad (x, y \in [-1, 1]),$$

or in other words

$$F_{Tx,Ty}(\frac{1}{8}t) \geq \min\{F_{x,y}(t), F_{x,Tx}(t), F_{y,Ty}(t), F_{x,Ty}(t), F_{y,Tx}(t)\},$$

for every $x, y \in X$ *and* $t \geq 0$. *Also,* $0 \in X$ *has the bounded orbit, so* T *has a unique fixed point 0 in* X, *by Theorem 7.*

The following example due to Ume [20].

Example 5. *Let* $X = [-1, 1]$, $T : X \to X$ *and* $\varphi : [0, \infty) \to [0, \infty)$ *be mappings defined as follows:*

$$T(x) = \begin{cases} 0, & -1 \leq x < 0, \\ \frac{x}{1+x}, & 0 \leq x < \frac{4}{5} \text{ or } \frac{7}{8} < x \leq 1, \\ \frac{-1}{16}x, & \frac{4}{5} \leq x \leq \frac{7}{8}, \end{cases} \qquad \varphi(t) = \begin{cases} t - \frac{t^2}{8}, & 0 \leq t \leq 1, \\ \frac{7}{8}t, & 1 < t. \end{cases}$$

Let $F_{x,y}(t) = \epsilon_0(t - \mid x - y \mid)$ *for all* $t \in \mathbb{R}$ *and* $x, y \in X$. *Then* (X, F, Δ_m) *is a complete probabilistic Menger space. It is easy to see that all of the assumptions of Theorem 7 are satisfied, and so* T *has a unique fixed point* ($x = 0$ *is a unique fixed point of* T). *On the other hand, we can show that* T *does not satisfy* (1).

Lemma 8. *[7] Let X be a nonempty set and $f : X \to X$ a mapping. Then there exists a subset $E \subseteq X$ such that $f(E) = f(X)$ and $f : E \to X$ is one-to-one.*

Theorem 9. *Let (X, F, Δ) be a complete Menger space under a continuous t-norm Δ and let self mappings T and S satisfy the following condition:*

$$F_{Tx,Ty}(\varphi(t)) \geq \min\{F_{Sx,Sy}(t), F_{Sx,Tx}(t), F_{Sy,Ty}(t), F_{Sx,Ty}(t), F_{Tx,Sy}(t)\},$$

for all $x, y \in X$, where $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ is a mapping the same as in Theorem 7. If $TX \subseteq SX$ and SX is a complete subset of X, then T and S have a unique coincidence point in X. Moreover, if T and S are weakly compatible (i.e, they commut at their coincidence points), then T and S have a unique common fixed point.

Proof. By Lemma 8, there exists $E \subseteq X$ such that $SE = SX$ and $S : E \to X$ is one-to-one. Now, define a mapping $U : SE \to SE$ by $U(Sx) = Tx$. Since S is one to one on E, U is well defined. Also we have

$$\begin{aligned} F_{U(Sx),U(Sy)}(\varphi(t)) &= F_{Tx,Ty}(\varphi(t)) \\ &\geq \min\{F_{Sx,Sy}(t), F_{Sx,Tx}(t), F_{Sy,Ty}(t), F_{Sx,Ty}(t), F_{Tx,Sy}(t)\}, \end{aligned}$$

for all $Sx, Sy \in SE$. Since $SE = SX$ is complete, by using Theorem 7, there exists $x^* \in X$ such that $U(Sx^*) = Sx^*$. Then $Tx^* = Sx^*$, and so T and S have a coincidence point, which is also unique.

If T and S are weakly compatible, since $Tx^* = Sx^*$, then we have

$$T(Tx^*) = TSx^* = STx^* = S(Sx^*).$$

Thus, $Tx^* = Sx^*$ is also a confidence point of T and S. By uniqueness of coincidence point of T and S, we get $Tx^* = Sx^* = x^*$.

Theorem 10. *Let (X, F, Δ) be a complete probabilistic Menger space under a continuous t-norm Δ. Suppose that $T : X \to X$ is a mapping satisfy the following condition:*

$$F_{Tx,Ty}(\alpha(t)t) \geq \min\{F_{x,y}(t), F_{x,Tx}(t), F_{y,Ty}(t), F_{x,Ty}(t), F_{Tx,y}(t)\}, \tag{12}$$

for all $t > 0$ and $x, y \in X$, where $\alpha : (0, \infty) \to [0, 1)$ is strictly decreasing function. If there exists $x_0 \in X$ with the bounded orbit, then there is a unique $x^* \in X$ such that $Tx^* = x^*$. Moreover, $(T^n x_0)$ converges to x^*.

Proof. Set $\varphi(t) = \alpha(t)t$, it is sufficient to prove that φ satisfying the hypothesis of Theorem 7. In fact, since $\alpha(t) < 1$, then $\varphi(t) < t$, for all $t > 0$. On the other hand, for all $n \in \mathbb{Z}^+$, we see that $0 \leq \varphi^{n+1}(r) = \varphi(\varphi^n(r)) < \varphi^n(r)$, thus the sequence $\{\varphi^n(r)\}$ is convergent for each $r > 0$. Let $\lim\limits_{n\to\infty} \varphi^n(r) = a \geq 0$, then $\lim\limits_{t\to a^+} \varphi(t) = a$. Suppose that $a > 0$, then by the monotony of α, we have

$$a = \lim_{t\to a^+} \varphi(t) = \lim_{t\to a^+} \alpha(t)t \leq \lim_{t\to a^+} \alpha\left(\frac{a}{2}\right) t = \alpha\left(\frac{a}{2}\right) a < a.$$

This is a contradiction. Thus $\lim\limits_{n\to\infty} \varphi^n(r) = 0$. Then by Theorem 7, the result follows.

Example 6. *Consider $X = [0,3]$ and define $F_{x,y}(t) = \epsilon_0(t- \mid x-y \mid)$ for all $x, y \in X$. Then (X, F, Δ_m) is a complete probabilistic Menger space. Let $\varphi(t) = \frac{t}{2}$, define continuous self mappings S and T on X as*

$$Tx = \frac{1}{6}x + 1, \qquad Sx = \frac{1}{3}(x + \frac{12}{5}), \qquad (x \in X).$$

Thus we have

$$\begin{aligned} F_{Tx,Ty}(\varphi(t)) = \epsilon_0\left(\frac{t}{2} - \frac{1}{6} \mid x-y \mid\right) &= \epsilon_0\left(t - \frac{1}{3} \mid x-y \mid\right) \\ &= F_{Sx,Sy}(t). \end{aligned}$$

It is easy to see that $TX \subseteq SX$, T and S are weakly compatible. Hence, we conclude that all the conditions of Theorem 9 hold, so T and S have a unique common fixed point $\frac{6}{5}$ in X.

Example 7. *Let $X = [0,\infty)$ and*

$$F_{x,y}(t) = \begin{cases} \frac{t}{t+|x-y|}, & t \leq \mid x-y \mid, \\ 1, & t > \mid x-y \mid. \end{cases}$$

Then (X, F, Δ_m) is a complete probabilistic Menger space. Define $Tx = \frac{x}{1+x}$ and $\alpha(t) = \frac{1}{1+t}$. By definition of T we have

$$\mid Tx - Ty \mid = \frac{\mid x-y \mid}{1+ \mid x-y \mid +2\min\{x,y\} + xy} \leq \frac{\mid x-y \mid}{1+ \mid x-y \mid}.$$

Clearly, if $\alpha(t)t > \mid Tx - Ty \mid$, then (12) *holds. Suppose now that $\alpha(t)t \leq \mid Tx - Ty \mid$. Then we have*

$$\frac{t}{1+t} \leq \mid Tx - Ty \mid \leq \frac{\mid x-y \mid}{1+ \mid x-y \mid},$$

so $t \leq\mid x-y \mid$ *and by definition of* F *we get*

$$\begin{aligned} F_{Tx,Ty}(\alpha(t)t) &= \frac{\alpha(t)t}{\alpha(t)t+\mid Tx-Ty\mid} \\ &= \frac{t}{t+(1+t)\mid Tx-Ty\mid} \\ &\geq \frac{t}{t+(1+t)\frac{|x-y|}{1+|x-y|}} \\ &\geq \frac{t}{t+(1+\mid x-y\mid)\frac{|x-y|}{1+|x-y|}} \\ &= \frac{t}{t+\mid x-y\mid} \\ &= F_{x,y}(t). \end{aligned}$$

Thus we proved that T *satisfies* (12). *Therefore, we showed that the mapping* T *satisfies all hypotheses of Theorem 10 and has a unique fixed point* 0.

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