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SOME CONNECTIONS BETWEEN VARIOUS SUBCLASSES OF HARMONIC UNIVALENT FUNCTIONS INVOLVING PASCAL DISTRIBUTION SERIES

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Abstract. In the present paper, we investigate connections between various subclasses of harmonic univalent functions by using a convolution operator involving the Pascal distribution series.

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1. Introduction

Let $\mathcal{H}$ denote the family of continuous complex valued harmonic functions of the form $f = h + \overline{g}$ defined in the open unit disk $\mathfrak{U} = \{z : |z| < 1\}$, where

$$h(z) = z + \sum_{n=2}^{\infty} a_n z^n \text{ and } g(z) = \sum_{n=1}^{\infty} b_n z^n \tag{1}$$

are analytic in $\mathfrak{U}$.
A necessary and sufficient condition for f to be locally univalent and sense-preserving in $\mathfrak{U}$ is that $|h'(z)| > |g'(z)|$ in $\mathfrak{U}$ (see [4]).
Denote by $\mathcal{SH}$ the subclass of $\mathcal{H}$ consisting of functions $f = h+\overline{g}$ which are harmonic, univalent and sense-preserving in $\mathfrak{U}$ and normalized by $f(0) = f_z(0) - 1 = 0$. One can easily show that the sense-preserving property implies that $|b_1| < 1$. The subclass $\mathcal{SH}^0$ of $\mathcal{SH}$ consisting of all functions in $\mathcal{SH}$ which have the additional property $b_1 = 0$. Note that $\mathcal{SH}$ reduces to the class $\mathcal{S}$ of normalized analytic univalent functions in $\mathfrak{U}$, if the co-analytic part of f is identically zero.
A function $f \in \mathcal{SH}$ is said to be harmonic starlike of order α $(0 \le \alpha < 1)$ in $\mathfrak{U}$ if and only if

$$\Re\left\{\frac{zf_z(z) - \bar{z}f_{\bar{z}}(z)}{f(z)}\right\} > \alpha, \quad (z \in \mathfrak{U}) \tag{2}$$

and is said to be harmonic convex of order α $(0 \le \alpha < 1)$ in $\mathfrak{U}$ if and only if

$$\Re\left\{\frac{z^2 f_{zz}(z) + z f_z(z) + \bar{z}^2 f_{\bar{z}\bar{z}}(z) + \bar{z} f_{\bar{z}}(z)}{z f_z(z) - \bar{z} f_{\bar{z}}(z)}\right\} > \alpha, \quad (z \in \mathfrak{U}). \tag{3}$$

These classes represented by $\mathcal{SH}^*(\alpha)$ and $\mathcal{KH}(\alpha)$, respectively, were extensively studied by Jahangiri [8]. Denote by $\mathcal{SH}^*$ and $\mathcal{KH}$ the classes $\mathcal{SH}^*(0)$ and $\mathcal{KH}(0)$, respectively. For definitions and properties of these classes, one may refer to [9, 10] or [3].

The elementary distributions such as the Poisson, the Pascal, the Logarithmic, the Binomial have been partially studied in the Geometric Function Theory from a theoretical point of view (see [1, 2, 5, 7]).

Let us consider a non-negative discrete random variable $\mathcal{X}$ with a Pascal probability generating function

$$P(\mathcal{X} = n) = \binom{n+r-1}{r-1} p^n (1-p)^r, \quad n \in \{0, 1, 2, 3, ...\}$$

where p, r are called the parameters.

Now we introduce a power series whose coefficients are probabilities of the Pascal distribution, that is

$$P_p^r(z) = z + \sum_{n=2}^{\infty} \binom{n+r-2}{r-1} p^{n-1} (1-p)^r z^n. \quad (r \ge 1,\ 0 \le p \le 1,\ z \in \mathfrak{U}) \tag{4}$$

Note that, by using ratio test we conclude that the radius of convergence of the above power series is $1/p$. Now, for $r, s \ge 1$ and $0 \le p, q \le 1$, we introduce the operator $P_{p,q}^{r,s} : \mathcal{H} \to \mathcal{H}$ by

$$P_{p,q}^{r,s}(f)(z) = P_p^r(z) * h(z) + \overline{P_q^s(z) * g(z)} = H(z) + \overline{G(z)}$$

where

$$\begin{aligned} H(z) &= z + \sum_{n=2}^{\infty} \binom{n+r-2}{r-1} p^{n-1} (1-p)^r a_n z^n \\ G(z) &= b_1 z + \sum_{n=2}^{\infty} \binom{n+s-2}{s-1} q^{n-1} (1-q)^s b_n z^n \end{aligned} \tag{5}$$

and "$*$" denotes the convolution (or Hadamard product) of power series.

2. Preliminary Lemmas

To prove our theorems we will use the following lemmas.

Lemma 1. *(See [6]) If $f = h+\overline{g} \in \mathcal{KH}^0$ where h and g are given by* (1) *with $b_1 = 0$, then*

$$|a_n| \leq \frac{n+1}{2}, \quad |b_n| \leq \frac{n-1}{2}.$$

Lemma 2. *(See [8]) Let $f = h+\overline{g}$ be given by* (1). *If for some α $(0 \leq \alpha < 1)$ and the inequality*

$$\sum_{n=2}^{\infty} (n-\alpha)|a_n| + \sum_{n=1}^{\infty} (n+\alpha)|b_n| \leq 1-\alpha \tag{6}$$

is hold, then f is harmonic, sense-preserving, univalent in $\mathfrak{U}$ and $f \in \mathcal{SH}^*(\alpha)$.

Define $\mathcal{TSH}^*(\alpha) = \mathcal{SH}^*(\alpha) \cap \mathcal{T}^2$ and $\mathcal{TKH}(\alpha) = \mathcal{KH}(\alpha) \cap \mathcal{T}^1$ where $\mathcal{T}^k$, $(k = 1,\ 2)$ consisting of the functions $f = h+\overline{g}$ in SH so that $h(z)$ and $g(z)$ are of the form

$$h(z) = z - \sum_{n=2}^{\infty} |a_n| z^n,\ g(z) = (-1)^k \sum_{n=1}^{\infty} |b_n| z^n,\ |b_1| < 1\ (k = 1,\ 2). \tag{7}$$

Remark 1. *(See [8]) Let $f = h+\overline{g}$ be given by* (7). *Then $f \in \mathcal{TSH}^*(\alpha)$ if and only if the coefficient condition* (6) *is satisfied. Also, if $f \in \mathcal{TSH}^*(\alpha)$, then*

$$|a_n| \leq \frac{1-\alpha}{n-\alpha}, \quad n \geq 2, \quad |b_n| \leq \frac{1-\alpha}{n+\alpha}, \quad n \geq 1. \tag{8}$$

Lemma 3. *(See [8]) Let $f = h+\overline{g}$ be given by* (1). *If for some α $(0 \leq \alpha < 1)$ and the inequality*

$$\sum_{n=2}^{\infty} n(n-\alpha)|a_n| + \sum_{n=1}^{\infty} n(n+\alpha)|b_n| \leq 1-\alpha \tag{9}$$

is hold, then f is harmonic, sense-preserving, univalent in $\mathfrak{U}$ and $f \in \mathcal{KH}(\alpha)$.

Remark 2. *(See [8]) Let $f = h+\overline{g}$ be given by* (7). *Then $f \in \mathcal{TKH}(\alpha)$ if and only if the coefficient condition* (9) *holds. Also, if $f \in \mathcal{TKH}(\alpha)$, then*

$$|a_n| \leq \frac{1-\alpha}{n(n-\alpha)}, \quad n \geq 2, \quad |b_n| \leq \frac{1-\alpha}{n(n+\alpha)}, \quad n \geq 1. \tag{10}$$

Lemma 4. *(See [6]) If $f = h+\overline{g} \in \mathcal{SH}^{*,0}$ where h and g are given by* (1) *with $b_1 = 0$, then*

$$|a_n| \leq \frac{(2n+1)(n+1)}{6}, \quad |b_n| \leq \frac{(2n-1)(n-1)}{6},\ n \geq 2.$$

3. Main Results

Theorem 5. *Let $r, s \geq 1$ and $0 \leq p, q < 1$. Also, let $f = h + \overline{g} \in \mathcal{H}$ is given by (1). If the inequalities*

$$\sum_{n=2}^{\infty} |a_n| + \sum_{n=1}^{\infty} |b_n| \leq 1, \quad (|b_1| < 1) \tag{11}$$

and

$$(1-p)^r + (1-q)^s \geq 1 + |b_1| + \frac{rp}{1-p} + \frac{sq}{1-q} \tag{12}$$

are hold, then the operator $P_{p,q}^{r,s}$ is harmonic, sense-preserving, univalent and maps $\mathcal{H}$ into $\mathcal{SH}^*$.

Proof. Note that $P_{p,q}^{r,s}(f) = H(z) + \overline{G(z)}$, where $H(z)$ and $G(z)$ are given by (5). To prove that $P_{p,q}^{r,s}(f)$ is locally univalent and sense-preserving it suffices to prove that $|H'(z)| - |G'(z)| > 0$ in $\mathfrak{U}$. Using (11), we compute

$$\begin{aligned}
\left|H'(z)\right| - \left|G'(z)\right| &> 1 - \sum_{n=2}^{\infty} n \binom{n+r-2}{r-1} p^{n-1} (1-p)^r \\
&\quad - |b_1| - \sum_{n=2}^{\infty} n \binom{n+s-2}{s-1} q^{n-1} (1-q)^s \\
&= 1 - |b_1| - \sum_{n=2}^{\infty} (n-1+1) \binom{n+r-2}{r-1} p^{n-1} (1-p)^r \\
&\quad - \sum_{n=2}^{\infty} (n-1+1) \binom{n+s-2}{s-1} q^{n-1} (1-q)^s \\
&= 1 - |b_1| - rp(1-p)^r \sum_{n=2}^{\infty} \binom{n+r-2}{r} p^{n-2} \\
&\quad - (1-p)^r \sum_{n=2}^{\infty} \binom{n+r-2}{r-1} p^{n-1} - sq(1-q)^s \sum_{n=2}^{\infty} \binom{n+s-2}{s} q^{n-2} \\
&\quad - (1-q)^s \sum_{n=2}^{\infty} \binom{n+s-2}{s-1} q^{n-1} \\
&= 1 - |b_1| - rp(1-p)^r \sum_{n=0}^{\infty} \binom{n+r}{r} p^{n} \\
&\quad - (1-p)^r \sum_{n=0}^{\infty} \binom{n+r-1}{r-1} p^{n} + (1-p)^r
\end{aligned}$$

$$\begin{aligned}
&-sq\left(1-q\right)^{s}\sum_{n=0}^{\infty}\binom{n+s}{s}q^{n}\\
&-\left(1-q\right)^{s}\sum_{n=0}^{\infty}\binom{n+s-1}{s-1}q^{n}+\left(1-q\right)^{s}\\
=\;&\left(1-p\right)^{r}+\left(1-q\right)^{s}-1-\left|b_1\right|-\frac{rp}{1-p}-\frac{sq}{1-q}\geq 0.
\end{aligned}$$

To prove $P_{p,q}^{r,s}(f)$ is univalent in $\mathfrak{U}$, referring Theorem 1 in [8], for $z_1 \neq z_2$ in $\mathfrak{U}$, we need to show that

$$\Re\frac{P_{p,q}^{r,s}(f)\left(z_2\right)-P_{p,q}^{r,s}(f)\left(z_1\right)}{z_2-z_1}>\int_0^1\left(\Re(H'\left(z\left(t\right)\right))-\left|G'\left(z\left(t\right)\right)\right|\right)dt. \tag{13}$$

By (11), we have

$$\begin{aligned}
\Re(H'\left(z\left(t\right)\right))-\left|G'\left(z\left(t\right)\right)\right| \;>\;& 1-\sum_{n=2}^{\infty}n\binom{n+r-2}{r-1}p^{n-1}\left(1-p\right)^{r}\\
&-\left|b_1\right|-\sum_{n=2}^{\infty}n\binom{n+s-2}{s-1}q^{n-1}\left(1-q\right)^{s}.
\end{aligned}$$

Using (12), we obtain that the inequality above is nonnegative. Therefore, from the inequality (13) we have

$$\Re\frac{P_{p,q}^{r,s}(f)\left(z_2\right)-P_{p,q}^{r,s}(f)\left(z_1\right)}{z_2-z_1}>0.$$

Hence univalency of $P_{p,q}^{r,s}(f)$ is proved.
In order to show that $P_{p,q}^{r,s}(f)\in\mathcal{SH}^{*}$, we need to prove $\Phi_1\leq 1$, by Lemma 2, where

$$\Phi_1=\sum_{n=2}^{\infty}n\binom{n+r-2}{r-1}p^{n-1}\left(1-p\right)^{r}\left|a_n\right|+\left|b_1\right|+\sum_{n=2}^{\infty}n\binom{n+s-2}{s-1}q^{n-1}\left(1-q\right)^{s}\left|b_n\right|.$$

Since $|a_n|\leq 1$, $|b_n|\leq 1$, $\forall n\geq 2$ because of (11), we have

$$\begin{aligned}
\Phi_1 \;\leq\;& rp\left(1-p\right)^{r}\sum_{n=0}^{\infty}\binom{n+r}{r}p^{n}+\left(1-p\right)^{r}\sum_{n=0}^{\infty}\binom{n+r-1}{r-1}p^{n}\\
&-\left(1-p\right)^{r}+\left|b_1\right|+sq\left(1-q\right)^{s}\sum_{n=0}^{\infty}\binom{n+s}{s}q^{n}
\end{aligned}$$

$$
\begin{aligned}
& + (1-q)^s \sum_{n=0}^{\infty} \binom{n+s-1}{s-1} q^n - (1-q)^s \\
= \; & |b_1| + \frac{rp}{1-p} + 1 - (1-p)^r + \frac{sq}{1-q} + 1 - (1-q)^s \\
\leq \; & 1
\end{aligned}
$$

from (12). Thus proof of Theorem 5 is complete.

Theorem 6. *Let $0 \leq \alpha < 1$, $r, s \geq 1$ and $0 \leq p, q < 1$. If the inequality*

$$
\begin{aligned}
& \frac{r(r+1)p^2}{(1-p)^2} + \frac{(4-\alpha)rp}{1-p} + \frac{s(s+1)q^2}{(1-q)^2} + \frac{(2+\alpha)sq}{1-q} \\
& \leq 2(1-\alpha)(1-p)^r
\end{aligned}
$$

is hold, then $P_{p,q}^{r,s}\left(\mathcal{KH}^0\right) \subset \mathcal{SH}^{*,0}(\alpha)$.

Proof. Suppose that $f = h + \overline{g} \in \mathcal{KH}^0$ where h and g are given by (1) with $b_1 = 0$. It suffices to show that $P_{p,q}^{r,s}(f) = H + \overline{G} \in \mathcal{SH}^{*,0}(\alpha)$, where H and G are given by (5) with $b_1 = 0$ in $\mathfrak{U}$. Using Lemma 2, we need to prove that $\Phi_2 \leq 1 - \alpha$, where

$$
\Phi_2 = \sum_{n=2}^{\infty} (n-\alpha) \binom{n+r-2}{r-1} (1-p)^r p^{n-1} |a_n| \tag{14}
$$

$$
+ \sum_{n=2}^{\infty} (n+\alpha) \binom{n+s-2}{s-1} (1-q)^s q^{n-1} |b_n|. \tag{15}
$$

Using Lemma 1, we compute

$$
\begin{aligned}
\Phi_2 \leq \; & \frac{1}{2} \left\{ \sum_{n=2}^{\infty} (n-\alpha)(n+1) \binom{n+r-2}{r-1} (1-p)^r p^{n-1} \right. \\
& \left. + \sum_{n=2}^{\infty} (n+\alpha)(n-1) \binom{n+s-2}{s-1} (1-q)^s q^{n-1} \right\} \\
= \; & \frac{1}{2} \left\{ \sum_{n=2}^{\infty} \left[(n-1)(n-2) + (4-\alpha)(n-1) + 2(1-\alpha)\right] \binom{n+r-2}{r-1} (1-p)^r p^{n-1} \right. \\
& \left. + \sum_{n=2}^{\infty} \left[(n-1)(n-2) + (2+\alpha)(n-1)\right] \binom{n+s-2}{s-1} (1-q)^s q^{n-1} \right\}
\end{aligned}
$$

$$
\begin{aligned}
&= \frac{1}{2}\left\{ r\left(r+1\right)p^{2}\left(1-p\right)^{r}\sum_{n=3}^{\infty}\binom{n+r-2}{r+1}p^{n-3}\right. \\
&\quad +\left(4-\alpha\right)rp\left(1-p\right)^{r}\sum_{n=2}^{\infty}\binom{n+r-2}{r}p^{n-2} \\
&\quad +2\left(1-\alpha\right)\left(1-p\right)^{r}\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}p^{n-2} \\
&\quad +s\left(s+1\right)q^{2}\left(1-q\right)^{s}\sum_{n=3}^{\infty}\binom{n+s-2}{s+1}q^{n-3} \\
&\quad \left. +\left(2+\alpha\right)sq\left(1-q\right)^{s}\sum_{n=2}^{\infty}\binom{n+s-2}{s}q^{n-2}\right\} \\
&= \frac{1}{2}\left\{ r\left(r+1\right)p^{2}\left(1-p\right)^{r}\sum_{n=0}^{\infty}\binom{n+r+1}{r+1}p^{n}\right. \\
&\quad +\left(4-\alpha\right)rp\left(1-p\right)^{r}\sum_{n=0}^{\infty}\binom{n+r}{r}p^{n} \\
&\quad +2\left(1-\alpha\right)\left(1-p\right)^{r}\sum_{n=0}^{\infty}\binom{n+r-1}{r-1}p^{n}-2\left(1-\alpha\right)\left(1-p\right)^{r} \\
&\quad +s\left(s+1\right)q^{2}\left(1-q\right)^{s}\sum_{n=0}^{\infty}\binom{n+s+1}{s+1}q^{n} \\
&\quad \left. +\left(2+\alpha\right)sq\left(1-q\right)^{s}\sum_{n=0}^{\infty}\binom{n+s}{s}q^{n}\right\} \\
&= \frac{1}{2}\left\{ \frac{r\left(r+1\right)p^{2}}{\left(1-p\right)^{2}}+\frac{\left(4-\alpha\right)rp}{1-p}+2\left(1-\alpha\right)\right. \\
&\quad \left. -2\left(1-\alpha\right)\left(1-p\right)^{r}+\frac{s\left(s+1\right)q^{2}}{\left(1-q\right)^{2}}+\frac{\left(2+\alpha\right)sq}{1-q}\right\}.
\end{aligned}
$$

The last expression is bounded above by $(1-\alpha)$ by the given condition.

Thus the proof of Theorem 6 is completed.

Theorem 7. *Suppose* $0 \leq \alpha < 1$, $r, s \geq 1$ *and* $0 \leq p, q < 1$. *If the inequality*

$$\begin{aligned} & \frac{2r\left(r+1\right)\left(r+2\right)p^{3}}{\left(1-p\right)^{3}}+\frac{\left(15-2\alpha\right)r\left(r+1\right)p^{2}}{\left(1-p\right)^{2}}+\frac{\left(24-9\alpha\right)rp}{1-p} \\ & +\frac{2s\left(s+1\right)\left(s+2\right)q^{3}}{\left(1-q\right)^{3}}+\frac{\left(9+2\alpha\right)s\left(s+1\right)q^{2}}{\left(1-q\right)^{2}}+\frac{\left(6+3\alpha\right)sq}{1-q} \\ \leq\ & 6\left(1-\alpha\right)\left(1-p\right)^{r} \end{aligned} \tag{16}$$

is hold then $P_{p,q}^{r,s}\left(\mathcal{SH}^{*,0}\left(\alpha\right)\right)\subset\mathcal{SH}^{*,0}\left(\alpha\right)$.

Proof. Suppose $f = h + \overline{g} \in \mathcal{SH}^{*.0}(\alpha)$ where h and g are given by (1) with $b_1 = 0$. It suffices to show that $P_{p,q}^{r,s}(f) = H + \overline{G} \in \mathcal{SH}^{*,0}(\alpha)$ where H and G are given by (5) with $b_1 = 0$. By Lemma 2, we need to prove that $\Phi_2 \leq 1-\alpha$, where

$$\begin{aligned} \Phi_2 &= \sum_{n=2}^{\infty}\left(n-\alpha\right)\binom{n+r-2}{r-1}\left(1-p\right)^{r}p^{n-1}\left|a_n\right| \\ &\quad +\sum_{n=2}^{\infty}\left(n+\alpha\right)\binom{n+s-2}{s-1}\left(1-q\right)^{s}q^{n-1}\left|b_n\right|. \end{aligned}$$

Using Lemma 4, we have

$$\begin{aligned} \Phi_2 &\leq \frac{1}{6}\left\{\sum_{n=2}^{\infty}\left(n-\alpha\right)\left(2n+1\right)\left(n+1\right)\binom{n+r-2}{r-1}\left(1-p\right)^{r}p^{n-1}\right. \\ &\quad \left.+\sum_{n=2}^{\infty}\left(n+\alpha\right)\left(2n-1\right)\left(n-1\right)\binom{n+s-2}{s-1}\left(1-q\right)^{s}q^{n-1}\right\} \\ &= \frac{1}{6}\left\{2\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}\left(n-1\right)\left(n-2\right)\left(n-3\right)\left(1-p\right)^{r}p^{n-1}\right. \\ &\quad +\left(15-2\alpha\right)\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}\left(n-1\right)\left(n-2\right)\left(1-p\right)^{r}p^{n-1} \\ &\quad +\left(24-9\alpha\right)\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}\left(n-1\right)\left(1-p\right)^{r}p^{n-1} \\ &\quad +6\left(1-\alpha\right)\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}\left(1-p\right)^{r}p^{n-1} \\ &\quad +2\sum_{n=2}^{\infty}\binom{n+s-2}{s-1}\left(n-1\right)\left(n-2\right)\left(n-3\right)\left(1-q\right)^{s}q^{n-1} \end{aligned}$$

$$+\left(9+2\alpha\right)\sum_{n=2}^{\infty}\binom{n+s-2}{s-1}\left(n-1\right)\left(n-2\right)\left(1-q\right)^{s}q^{n-1}$$

$$\left.+\left(6+3\alpha\right)\sum_{n=2}^{\infty}\binom{n+s-2}{s-1}\left(n-1\right)\left(1-q\right)^{s}q^{n-1}\right\}$$

$$=\frac{1}{6}\left\{2r\left(r+1\right)\left(r+2\right)p^{3}\left(1-p\right)^{r}\sum_{n=4}^{\infty}\binom{n+r-2}{r+2}p^{n-4}\right.$$

$$+\left(15-2\alpha\right)r\left(r+1\right)p^{2}\left(1-p\right)^{r}\sum_{n=3}^{\infty}\binom{n+r-2}{r+1}p^{n-3}$$

$$+\left(24-9\alpha\right)rp\left(1-p\right)^{r}\sum_{n=2}^{\infty}\binom{n+r-2}{r}p^{n-2}$$

$$+6\left(1-\alpha\right)\sum_{n=2}^{\infty}\binom{n+r-2}{r-1}\left(1-p\right)^{r}p^{n-1}$$

$$+2s\left(s+1\right)\left(s+2\right)q^{3}\left(1-q\right)^{s}\sum_{n=4}^{\infty}\binom{n+s-2}{s+2}q^{n-4}$$

$$+\left(9+2\alpha\right)s\left(s+1\right)q^{2}\left(1-q\right)^{s}\sum_{n=3}^{\infty}\binom{n+s-2}{s+1}q^{n-3}$$

$$\left.+\left(6+3\alpha\right)sq\left(1-q\right)^{s}\sum_{n=2}^{\infty}\binom{n+s-2}{s}q^{n-2}\right\}$$

$$=\frac{1}{6}\left\{2r\left(r+1\right)\left(r+2\right)p^{3}\left(1-p\right)^{r}\sum_{n=0}^{\infty}\binom{n+r+2}{r+2}p^{n}\right.$$

$$+\left(15-2\alpha\right)r\left(r+1\right)p^{2}\left(1-p\right)^{r}\sum_{n=0}^{\infty}\binom{n+r+1}{r+1}p^{n}$$

$$+\left(24-9\alpha\right)rp\left(1-p\right)^{r}\sum_{n=0}^{\infty}\binom{n+r}{r}p^{n}$$

$$+6\left(1-\alpha\right)\left(1-p\right)^{r}\sum_{n=0}^{\infty}\binom{n+r-1}{r-1}p^{n}-6\left(1-\alpha\right)\left(1-p\right)^{r}$$

$$+2s\left(s+1\right)\left(s+2\right)q^{3}\left(1-q\right)^{s}\sum_{n=0}^{\infty}\binom{n+s+2}{s+2}q^{n}$$

$$+\left(9+2\alpha\right)s\left(s+1\right)q^{2}\left(1-q\right)^{s}\sum_{n=0}^{\infty}\binom{n+s+1}{s+1}q^{n}$$

$$
\begin{aligned}
& + (6+3\alpha)\, sq\,(1-q)^s \sum_{n=0}^{\infty} \binom{n+s}{s} q^n \Bigg\} \\
= \;& \frac{1}{6}\Bigg\{ \frac{2r\,(r+1)\,(r+2)\,p^3}{(1-p)^3} + \frac{(15-2\alpha)\, r\,(r+1)\,p^2}{(1-p)^2} \\
& + \frac{(24-9\alpha)\, rp}{1-p} + 6\,(1-\alpha) - 6\,(1-\alpha)\,(1-p)^r \\
& + \frac{2s\,(s+1)\,(s+2)\,q^3}{(1-q)^3} + \frac{(9+2\alpha)\, s\,(s+1)\,q^2}{(1-q)^2} + \frac{(6+3\alpha)\, sq}{1-q} \Bigg\} \\
\leq \;& 1-\alpha
\end{aligned}
$$

by the given condition.

Theorem 8. *If* $0 \leq \alpha < 1$*,* $r, s \geq 1$ *and* $0 \leq p, q < 1$ *then* $P_{p,q}^{r,s}\left(\mathcal{TSH}^*(\alpha)\right) \subset \mathcal{TSH}^*(\alpha)$ *if and only if the inequality*

$$
(1-p)^r + (1-q)^s \geq 1 + \frac{(1+\alpha)\,|b_1|}{(1-\alpha)}
$$

is hold.

Proof. Suppose $f = h + \overline{g} \in \mathcal{TSH}^*(\alpha)$ where h and g are given by (7). We need to prove that the operator

$$
\begin{aligned}
P_{p,q}^{r,s}(f)\,(z) \;=\;& z - \sum_{n=2}^{\infty} \binom{n+r-2}{r-1} (1-p)^r\, p^{n-1}\, |a_n|\, z^n \\
& + |b_1|\,\overline{z} + \sum_{n=2}^{\infty} \binom{n+s-2}{s-1} (1-q)^s\, q^{n-1}\, |b_n|\, \overline{z}^n
\end{aligned}
$$

is in $\mathcal{TSH}^*(\alpha)$ if and only if $\Phi_3 \leq 1-\alpha$, where

$$
\begin{aligned}
\Phi_3 \;=\;& \sum_{n=2}^{\infty} (n-\alpha) \binom{n+r-2}{r-1} (1-p)^r\, p^{n-1}\, |a_n| \\
& + (1+\alpha)\,|b_1| + \sum_{n=2}^{\infty} (n+\alpha) \binom{n+s-2}{s-1} (1-q)^s\, q^{n-1}\, |b_n| .
\end{aligned}
$$

By Remark 1, we have

$$
\begin{aligned}
\Phi_3 \;\leq\;& (1-\alpha) \Bigg\{ \sum_{n=2}^{\infty} \binom{n+r-2}{r-1} (1-p)^r\, p^{n-1} \\
& + \sum_{n=2}^{\infty} \binom{n+s-2}{s-1} (1-q)^s\, q^{n-1} \Bigg\} + (1+\alpha)\,|b_1|
\end{aligned}
$$

$$\begin{aligned}
&= (1-\alpha)\left\{(1-p)^r\sum_{n=0}^{\infty}\binom{n+r-1}{r-1}p^n-(1-p)^r\right.\\
&\quad\left.+(1-q)^s\sum_{n=0}^{\infty}\binom{n+s-1}{s-1}q^n-(1-q)^s\right\}+(1+\alpha)|b_1|\\
&= (1-\alpha)\{2-(1-p)^r-(1-q)^s\}+(1+\alpha)|b_1|\\
&\leq 1-\alpha
\end{aligned}$$

by the given condition and thus the proof of the theorem is completed.

We next explore a sufficient condition which guarantees that $P_{p,q}^{r,s}$ maps $\mathcal{KH}^0$ into $\mathcal{KH}^0(\alpha)$.

Theorem 9. *Suppose* $0 \leq \alpha < 1$*,* $r,s \geq 1$ *and* $0 \leq p,q < 1$*. If the inequality*

$$\begin{aligned}
&\frac{r(r+1)(r+2)p^3}{(1-p)^3}+\frac{(7-\alpha)r(r+1)p^2}{(1-p)^2}+\frac{(10-4\alpha)rp}{1-p}\\
&+\frac{s(s+1)(s+2)q^3}{(1-q)^3}+\frac{(5+\alpha)s(s+1)q^2}{(1-q)^2}+\frac{(4+2\alpha)sq}{1-q}\\
\leq\ & 2(1-\alpha)(1-p)^r
\end{aligned}$$

is hold, then $P_{p,q}^{r,s}\left(\mathcal{KH}^0\right)\subset\mathcal{KH}^0(\alpha)$*.*

Proof. Let $f = h+\overline{g} \in \mathcal{KH}^0$ where h and g are given by (1) with $b_1 = 0$. It suffices to show that $P_{p,q}^{r,s}(f) = H+\overline{G} \in \mathcal{KH}^0(\alpha)$ where H and G are given by (5) with $b_1 = 0$. Referring Lemma 1, we need to prove that $\Phi_4 \leq 1-\alpha$, where

$$\begin{aligned}
\Phi_4 &= \sum_{n=2}^{\infty}n(n-\alpha)\binom{n+r-2}{r-1}(1-p)^r p^{n-1}|a_n|\\
&\quad+\sum_{n=2}^{\infty}n(n+\alpha)\binom{n+s-2}{s-1}(1-q)^s q^{n-1}|b_n|.
\end{aligned}$$

Using Lemma 1, we have

$$\begin{aligned}
\Phi_4 &\leq \frac{1}{2}\left\{\sum_{n=2}^{\infty}(n-1)(n-2)(n-3)\binom{n+r-2}{r-1}(1-p)^r p^{n-1}\right.\\
&\quad+\sum_{n=2}^{\infty}(7-\alpha)(n-1)(n-2)\binom{n+r-2}{r-1}(1-p)^r p^{n-1}
\end{aligned}$$

$$
\begin{aligned}
& +\sum_{n=2}^{\infty}(10-4\alpha)(n-1)\binom{n+r-2}{r-1}(1-p)^r p^{n-1} \\
& +\sum_{n=2}^{\infty}2(1-\alpha)\binom{n+r-2}{r-1}(1-p)^r p^{n-1} \\
& +\sum_{n=2}^{\infty}(n-1)(n-2)(n-3)\binom{n+s-2}{s-1}(1-q)^s q^{n-1} \\
& +\sum_{n=2}^{\infty}(5+\alpha)(n-1)(n-2)\binom{n+s-2}{s-1}(1-q)^s q^{n-1} \\
& \left.+\sum_{n=2}^{\infty}(4+2\alpha)(n-1)\binom{n+s-2}{s-1}(1-q)^s q^{n-1}\right\} \\
= & \ \frac{1}{2}\left\{\frac{r(r+1)(r+2)p^3}{(1-p)^3}+\frac{(7-\alpha)r(r+1)p^2}{(1-p)^2}+\frac{(10-4\alpha)rp}{1-p}\right. \\
& +2(1-\alpha)-2(1-\alpha)(1-p)^r \\
& \left.+\frac{s(s+1)(s+2)q^3}{(1-q)^3}+\frac{(5+\alpha)s(s+1)q^2}{(1-q)^2}+\frac{(4+2\alpha)sq}{1-q}\right\} \\
\leq & \ 1-\alpha
\end{aligned}
$$

by the given condition.

The proofs of following theorems are similar to previous theorems so we omit them.

Theorem 10. *Let $0 \leq \alpha < 1, r, s \geq 1$ and $0 \leq p, q < 1$. If the inequality*

$$(1-p)^r+(1-q)^s \geq 1+\frac{rp}{1-p}+\frac{sq}{1-q}+\frac{(1+\alpha)}{(1-\alpha)}|b_1| \tag{17}$$

is hold, then $P_{p,q}^{r,s}(\mathcal{TSH}^*(\alpha)) \subset \mathcal{KH}(\alpha)$.

Theorem 11. *If $0 \leq \alpha < 1$, $r, s \geq 1$ and $0 \leq p, q < 1$ then $P_{p,q}^{r,s}(\mathcal{TKH}(\alpha)) \subset \mathcal{TKH}(\alpha)$ if and only if the inequality*

$$(1-p)^r+(1-q)^s \geq 1+\frac{(1+\alpha)|b_1|}{(1-\alpha)}$$

is hold.

Example 1. *Consider the harmonic polynomial $f_1(z) = z - \frac{1}{2}\bar{z}^2$. If we take $s = 10$ and $q = 0.1$ then from (5), we have*

$$P_{p,0.1}^{r,10}(f_1)(z) = z - 0.17\bar{z}^2.$$

One can easily see that coefficients of $f_1(z)$ satisfy condition (11). Condition (12) is also hold for $s = 10, q = 0.1$ and specific choices of r and p such as when $r = 1$ p can be chosen from 0 to 0.49 and when $r = 2$ p can be chosen from 0 to 0.31. Then, using Theorem 5, $P^{r,10}_{p,0.1}(f_1) \in \mathcal{SH}^*$. Images of concentric circles inside $\mathfrak{U}$ under the functions f_1 and $P^{r,10}_{p,0.1}(f_1)$ are shown in Figures 1 and 2.

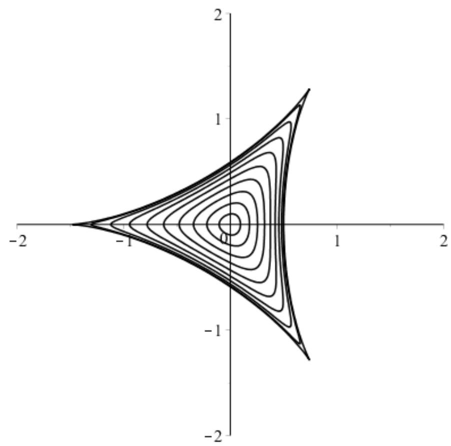

Figure 1: Image of f_1

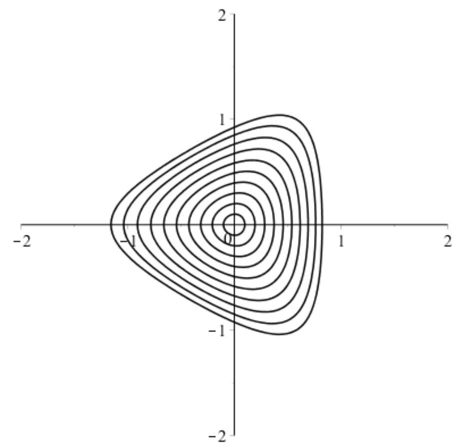

Figure 2: Image of $P^{r,10}_{p,0.1}(f_1)$

Example 2. *Consider the harmonic right half plane mapping $f_0(z) = \frac{z-\frac{1}{2}z^2}{(1-z)^2} + \frac{-\frac{1}{2}\overline{z}^2}{(1-\overline{z})^2} \in \mathcal{KH}^0$. If we take $r = 2$, $s = 2$, $p = 0.01$ and $q = 0.01$ then from (5), we have*

$$
\begin{aligned}
P^{2,2}_{0.01,\,0.01}(f_0)(z) &= z + \sum_{n=2}^{\infty} \frac{n(n+1)}{2}\,(0.01)^{n-1}(0.99)^2 z^n \\
&\quad + \sum_{n=2}^{\infty} \frac{n(-n+1)}{2}(0.01)^{n-1}(0.99)^2 \overline{z}^n.
\end{aligned}
$$

Then, according to the Theorem 9, $P^{2,2}_{0.01,\,0.01}(f_0)(z) \in \mathcal{KH}^0(\alpha)$ for $0 \le \alpha < 1$. Images of concentric circles inside $\mathfrak{U}$ under the functions f_0 and $P^{2,2}_{0.01,\,0.01}(f_0)$ are shown in Figures 3 and 4.

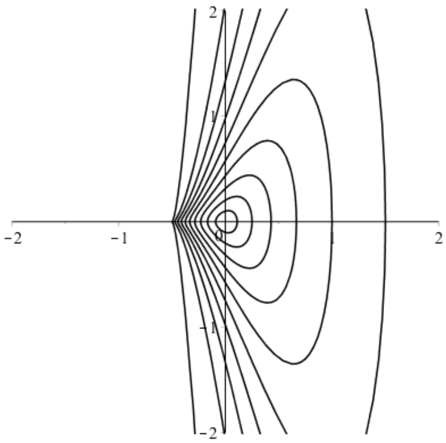

Figure 3: Image of f_0

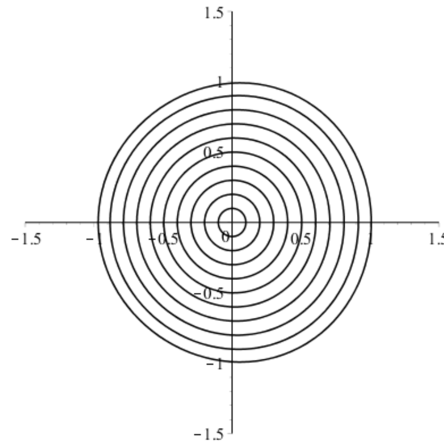

Figure 4: Image of $P^{2,2}_{0.01,0.01}(f_0)$

References

[1] Ş. Altınkaya, S. Yalçın, *Poisson distribution series for certain subclasses of starlike functions with negative coefficients*, Annals of Oradea University Mathematics Fascicola 2 24, (2017), 5-8.

[2] Ş. Altınkaya, S. Yalçın, *Poisson distribution series for analytic univalent functions*, Complex Analysis and Operator Theory 12, (2018), 1315-1319.

[3] Y. Avcı, E. Zlotkiewicz, *On harmonic univalent mappings*, Ann. Univ. Mariae Curie Sklodowska Sect. A 44, (2009), 1-7.

[4] J. Clunie, T. Sheil-Small, *Harmonic univalent functions*, Ann. Acad. Sci. Fenn. Ser. A I Math. 9, (1984), 3-25.

[5] S. Çakmak, S. Yalçın, Ş. Altınkaya, *Some connections between various classes of analytic functions associated with the power series distribution*, Sakarya University Journal of Science 23, (2019), 982-985.

[6] P. Duren, *Harmonic Mappings in the Plane*, Cambridge University Press, Cambridge (2004).

[7] S. M. El-Deeb, T. Bulboaca, J. Dziok, *Pascal distribution series connected with certain subclasses of univalent functions*, Kyungpook Math. J. 59, (2019), 301-314.

[8] J. M. Jahangiri, *Harmonic functions starlike in the unit disk*, J. Math. Anal. Appl. 235, (1999), 470-477.

[9] H. Silverman, *Harmonic univalent function with negative coefficients*, J. Math. Anal. Appl. 220, (1998), 283-289.

[10] H. Silverman, E. M. Silvia, *Subclasses of harmonic univalent functions*, New Zealand J. Math. 28, (1999), 275-284.

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