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SUBCLASSES OF SPIRAL-LIKE FUNCTIONS ASSOCIATED WITH PASCAL DISTRIBUTION SERIES

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Abstract. The aim of the current paper is to obtain the sufficient conditions and inclusion relations for Pascal distribution series to be in some subclasses of Spiral-like functions in the open unit disk $\mathbb{U}$. Further, we study an integral operator related to Pascal distribution series, and some consequences and corollaries of the main results are also considered.

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1. Introduction and definitions

Let $\mathcal{H}$ denote the class of analytic functions in the open unit disc $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$ on the complex plane $\mathbb{C}$. Furthermore, let $\mathcal{A}$ denote the subclass of $\mathcal{H}$ comprising of functions f normalized by $f(0) = 0$, $f'(0) = 1$ and let $\mathcal{S} \subset \mathcal{A}$ denote the class of functions which are univalent in $\mathbb{U}$. Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k,\ z \in \mathbb{U}, \tag{1}$$

which are analytic in $\mathbb{U}$ and normalized by the conditions $f(0) = 0 = f'(0) - 1$. Also, denote by $\mathcal{T}$ the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{k=2}^{\infty} a_k z^k,\ z \in \mathbb{U},\ \text{with } a_k \geq 0,\ k \in \mathbb{N},\ k \geq 2. \tag{2}$$

For functions $f \in \mathcal{A}$ given by (1) and $F \in \mathcal{A}$ given by $F(z) = z + \sum\limits_{k=2}^{\infty} A_k z^k$, we recall that the reputed Hadamard Product (or convolution) of f and F is given by

$$(f * F)(z) := z + \sum_{k=2}^{\infty} a_k A_k z^k,\ z \in \mathbb{U}.$$

Let a function f be analytic and univalent in $\mathbb{U}$ on the complex plane $\mathbb{C}$ with the normalization $f(0) = 0$, then f maps $\mathbb{U}$ onto a starlike domain with respect to $w_0 = 0$ if and only if

$$\mathfrak{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > 0 \quad (z \in \mathbb{U}). \tag{3}$$

It is well known that if an analytic function f satisfies (3) and $f(0) = 0$, $f'(0) \neq 0$, then f is univalent and starlike in $\mathbb{U}$. This class is denoted by $\mathcal{S}^*$. Denote by $\mathcal{K}$ the reputed class of convex functions. It is an established fact that

$$f \in \mathcal{K} \Leftrightarrow zf'(z) \in \mathcal{S}^*.$$

We say that an analytic function f is subordinate to an analytic function F, and write $f(z) \prec F(z)$, if and only if there exists a function ν, analytic in $\mathbb{U}$ such that $\nu(0) = 0$, $|\nu(z)| < 1$ for $z \in \mathbb{U}$ and $f(z) = F(\nu(z))$. In particular, if F is univalent in $\mathbb{U}$, then we have the following equivalence:

$$f(z) \prec F(z) \iff f(0) = F(0) \text{ and } f(\mathbb{U}) \subset F(\mathbb{U}).$$

For arbitrary fixed numbers A, B; $-1 \leq B < A \leq 1$; denote by $P(A; B)$ the family of functions $\Psi(z) = 1 + c_1 z + c_2 z^2 + \cdots$; regular in $\mathbb{U}$ such that $\Psi \in P(A; B)$ if and only if $\Psi(z) \prec \frac{1+Az}{1+Bz}$ for every $z \in \mathbb{U}$. This class was introduced by Janowski [12]. We remember the following notions subclass of spiral-like functions and subclasses of spirallike convex functions due to Robertson [20].

For $|\lambda| < \frac{\pi}{2}$ and $-1 \leq B < A \leq 1$; a function $f \in \mathcal{A}$ is said to be in the class of:
(i) λ–spirallike functions, denoted by $\mathcal{S}^\lambda(A, B)$, if it satisfies the condition

$$1 + \frac{e^{i\lambda}}{cos\lambda}\left(\frac{zf'(z)}{f(z)} - 1\right) \prec \frac{1 + Az}{1 + Bz} \quad z \in \mathbb{U}, \tag{4}$$

and
(ii) λ–spirallike convex functions, denoted by $\mathcal{K}^\lambda(A, B)$, if it satisfies the condition

$$1 + \frac{e^{i\lambda}}{cos\lambda}\left(\frac{zf''(z)}{f'(z)}\right) \prec \frac{1 + Az}{1 + Bz} \quad z \in \mathbb{U}. \tag{5}$$

From (4) and (5) it follows that

$$f \in \mathcal{K}^\lambda(A, B) \Leftrightarrow zf'(z)\mathcal{S}^\lambda(A, B).$$

A variable μ is said to be Pascal distribution if it takes the values $0, 1, 2, 3, \ldots$ with probabilities

$(1-\rho)^t$, $\frac{\rho t(1-\rho)^t}{1!}$, $\frac{\rho^2 t(t+1)(1-\rho)^t}{2!}$, $\frac{\rho^3 t(t+1)(t+2)(1-\rho)^t}{3!}$, ..., respectively, where ρ and t are called the parameters, and thus

$$P(\mu = k) = \binom{k+t-1}{t-1} \rho^k (1-\rho)^t, k = 0, 1, 2, 3, \ldots.$$

Very recently, El-Deeb [7] (see also [5, 15]) introduced a power series whose coefficients are probabilities of Pascal distribution, that is

$$\Phi_\rho^t(z) := z + \sum_{k=2}^{\infty} \binom{k+t-2}{t-1} \rho^{k-1} (1-\rho)^t z^k, \; z \in \mathbb{U}, \tag{6}$$

where $t \geq 1$, $0 \leq \rho \leq 1$, and we observe that, by ratio test the radius of convergence of above series is infinity.

Let consider the linear operator $\mathcal{I}_\rho^t : \mathcal{A} \to \mathcal{A}$ defined by the Hadamard product or convolution

$$\mathcal{I}_\rho^t f(z) := \Phi_\rho^t(z) * f(z) = z + \sum_{k=2}^{\infty} \binom{k+t-2}{t-1} \rho^{k-1} (1-\rho)^t a_k z^k, \; z \in \mathbb{U},$$

where $t \geq 1$ and $0 \leq \rho \leq 1$.

Motivated by several earlier results on connections between various subclasses of analytic and univalent functions, by using hypergeometric functions (see for example, [4, 10, 13, 21, 22]) and by the recent investigations (see, for example [1, 2, 3, 9, 14, 16, 17, 18, 19]), in the present paper we determine the sufficient conditions for Φ_ρ^t to be in our classes $\mathcal{S}^\lambda(A, B)$ and $\mathcal{K}^\lambda(A, B)$. We give connections of these subclasses with $\mathcal{R}^\tau(A, B)$, and finally, we give sufficient conditions for the function f such that its image by the integral operator $\mathcal{G}_\rho^t f(z) = \int_0^z \frac{\mathcal{I}_\rho^t f(\eta)}{\eta} d\eta$ belongs to the above classes.

To prove our main results, we need the following lemmas.

Lemma 1. *(See [6, Corollary 3.2]) A function f of the form* (2) *is in the class $\mathcal{S}^\lambda(A, B)$ if*

$$\sum_{k=2}^{\infty} \left[(k-1)(1-B) + (A-B)cos\lambda\right] |a_k| \leq (A-B)cos\lambda. \tag{7}$$

Lemma 2. *(See [6, Corollary 3.4]) A function f of the form* (2) *is in the class $\mathcal{K}^\lambda(A, B)$ if*

$$\sum_{k=2}^{\infty} k\left[(k-1)(1-B) + (A-B)cos\lambda\right] |a_k| \leq (A-B)cos\lambda.$$

2. Sufficient conditions for $\Phi_\rho^t \in \mathcal{S}^\lambda(A,B)$ and $\Phi_\rho^t \in \mathcal{K}^\lambda(A,B)$

For convenience throughout in the sequel, we use the following identities that hold at least for $t \geq 1$ and $0 \leq \rho < 1$:

$$\sum_{k=0}^{\infty}\binom{k+t-1}{t-1}\rho^k = \frac{1}{(1-\rho)^t}, \quad \sum_{k=0}^{\infty}\binom{k+t-2}{t-2}\rho^k = \frac{1}{(1-\rho)^{t-1}},$$
$$\sum_{k=0}^{\infty}\binom{k+t}{t}\rho^k = \frac{1}{(1-\rho)^{t+1}}, \quad \sum_{k=0}^{\infty}\binom{k+t+1}{t+1}\rho^k = \frac{1}{(1-\rho)^{t+2}},$$

By simple calculations, we derive the following relations:

$$\sum_{k=2}^{\infty}\binom{k+t-2}{t-1}\rho^{k-1} = \sum_{k=0}^{\infty}\binom{k+t-1}{t-1}\rho^k - 1 = \frac{1}{(1-\rho)^t} - 1,$$
$$\sum_{k=2}^{\infty}(k-1)\binom{k+t-2}{t-1}\rho^{k-1} = \rho t\sum_{k=0}^{\infty}\binom{k+t}{t}\rho^k = \frac{\rho t}{(1-\rho)^{t+1}},$$
$$\sum_{k=3}^{\infty}(k-1)(k-2)\binom{k+t-2}{t-1}\rho^{k-1} = \rho^2 t(t+1)\sum_{k=0}^{\infty}\binom{k+t+1}{t+1}\rho^k$$
$$= \frac{\rho^2 t(t+1)}{(1-\rho)^{t+2}}.$$

Unless otherwise mentioned, we shall assume in this paper that $|\lambda| < \frac{\pi}{2}$ and $-1 \leq B < A \leq 1$, while $t \geq 2$ and $0 \leq \rho < 1$.

In the first two results we obtain the sufficient conditions for $\Phi_\rho^t \in \mathcal{S}^\lambda(A,B)$ and $\Phi_\rho^t \in \mathcal{K}^\lambda(A,B)$, respectively.

Theorem 3. *We have $\Phi_\rho^t \in \mathcal{S}^\lambda(A,B)$ if*

$$\frac{(1-B)\rho t}{(1-\rho)^{t+1}} \leq (A-B)cos\lambda \tag{8}$$

Proof. Since Φ_ρ^t is defined by (6), in view of Lemma 1 it is sufficient to show that

$$\sum_{k=2}^{\infty}[(k-1)(1-B)+(A-B)cos\lambda]\binom{k+t-2}{t-1}\rho^{k-1}(1-\rho)^t \leq (A-B)cos\lambda. \tag{9}$$

Writing in the left hand side of (9) we get

$$\begin{aligned}
&\sum_{k=2}^{\infty}[(k-1)(1-B)+(A-B)cos\lambda]\binom{k+t-2}{t-1}\rho^{k-1}(1-\rho)^t \\
&=(1-B)\sum_{k=2}^{\infty}(k-1)\binom{k+t-2}{t-1}\rho^{k-1}(1-\rho)^t \\
&+(A-B)cos\lambda\sum_{k=2}^{\infty}\binom{k+t-2}{t-1}\rho^{k-1}(1-\rho)^t \\
&=\frac{(1-B)\rho t}{1-\rho}+(A-B)cos\lambda\left(1-(1-\rho)^t\right),
\end{aligned}$$

but the last expression is upper bounded by $(A-B)cos\lambda$ if (8) holds.

Theorem 4. *We have* $\Phi_\rho^t \in \mathcal{K}^\lambda(A,B)$*, if*

$$(1-B)\frac{\rho^2 t(t+1)}{(1-\rho)^{t+2}}+[(A-B)cos\lambda+2(1-B)]\frac{\rho t}{(1-\rho)^{t+1}}\leq(A-B)cos\lambda. \quad (10)$$

Proof. In view of Lemma 2 we must show that

$$\sum_{k=2}^{\infty}k\left[(k-1)(1-B)+(A-B)cos\lambda\right]\binom{k+t-2}{t-1}\rho^{k-1}(1-\rho)^t\leq(A-B)cos\lambda. \quad (11)$$

Substituting

$$\begin{aligned}
k^2&=(k-1)(k-2)+3(k-1)+1,\\
k&=(k-1)+1,
\end{aligned}$$

in (11), we get

$$\begin{aligned}
&\sum_{k=2}^{\infty}k\left[(k-1)(1-B)+(A-B)cos\lambda\right]\binom{k+t-2}{t-1}\rho^{k-1}(1-\rho)^t \\
&=\sum_{k=2}^{\infty}\left[(k^2-k)(1-B)+k(A-B)cos\lambda\right]\binom{k+t-2}{t-1}\rho^{k-1}(1-\rho)^t
\end{aligned}$$

$$= (1-B)\sum_{k=2}^{\infty}[(k-1)(k-2)+2(k-1)]\binom{k+jt-2}{t-1}\rho^{k-1}(1-\rho)^{j}$$
$$+(A-B)cos\lambda\sum_{k=2}^{\infty}[(k-1)+1]\binom{k+t-2}{t-1}\rho^{k-1}(1-\rho)^{t}$$
$$= (1-B)\frac{\rho^2 t(t+1)}{(1-\rho)^2}+[(A-B)cos\lambda+2(1-B)]\frac{\rho t}{1-\rho}$$
$$+(A-B)cos\lambda\left(1-(1-\rho)^t\right)$$

therefore, the last expression is upper bounded by $(A-B)cos\lambda$ if the inequality (10) is satisfied.

3. Sufficient conditions for $\mathcal{I}_\rho^t(\mathcal{R}^\tau(C,D)) \subset \mathcal{S}^\lambda(A,B)$ and $\mathcal{I}_\rho^t(\mathcal{R}^\tau(C,D)) \subset \mathcal{K}^\lambda(A,B)$

In [8] Dixit and Pal introduced the following subclass of $\mathcal{A}$:

Definition 1. *(See [8]) A function $f \in \mathcal{A}$ is said to be in the class $\mathcal{R}^\tau(C,D)$, with $\tau \in \mathbb{C}\setminus\{0\}$ and $-1 \leq D < C \leq 1$, if it satisfies the inequality*

$$\left|\frac{f'(z)-1}{(C-D)\tau - D[f'(z)-1]}\right| < 1,\ z \in \mathbb{U}.$$

Also, they proved the next sharp estimations regarding the coefficients of the power expansions for the functions belonging to this class, as follows:

Lemma 5. *(See [8]) If $f \in \mathcal{R}^\tau(C,D)$ is of the form (1), then*

$$|a_k| \leq (C-D)\frac{|\tau|}{k},\ k \in \mathbb{N}\setminus\{1\},$$

and the result is sharp.

Making use of Lemma 5, we will research the action of the Pascal distribution series on the class $\mathcal{S}^\lambda(A,B)$.

Theorem 6. *If $f \in \mathcal{R}^\tau(C,D)$, then $\mathcal{I}_\rho^t f \in \mathcal{S}^\lambda(A,B)$ if*

$$(C-D)|\tau|\left\{(1-B)\ \left(1-(1-\rho)^t\right)\right.$$
$$\left.-\frac{(1-B)-(A-B)cos\lambda}{\rho(t-1)}\left[(1-\rho)-(1-\rho)^t-\rho(t-1)(1-\rho)^t\right]\right\}$$
$$\leq (A-B)cos\lambda. \tag{12}$$

Proof. According to Lemma 1 it is sufficient to show that

$$\sum_{k=2}^{\infty}[(k-1)(1-B)+(A-B)cos\lambda]\binom{k+t-2}{t-1}\rho^{k-1}(1-\rho)^t\,|a_k|\leq(A-B)cos\lambda.$$

Since $f\in\mathcal{R}^{\tau}(C,D)$, using Lemma 5 we have

$$|a_k|\leq\frac{(C-D)\,|\tau|}{k},\ k\in\mathbb{N}\setminus\{1\},$$

therefore

$$\begin{aligned}
&\sum_{k=2}^{\infty}[(k-1)(1-B)+(A-B)cos\lambda]\binom{k+t-2}{t-1}\rho^{k-1}(1-\rho)^t\,|a_k|\\
&\leq(C-D)\,|\tau|\sum_{k=2}^{\infty}\frac{1}{k}[(k-1)(1-B)+(A-B)cos\lambda]\binom{k+t-2}{t-1}\rho^{k-1}(1-\rho)^t\\
&=(C-D)\,|\tau|\,(1-\rho)^t\Bigg[(1-B)\sum_{k=2}^{\infty}\binom{k+t-2}{t-1}\rho^{k-1}\\
&\qquad\qquad-[(1-B)-(A-B)cos\lambda]\sum_{k=2}^{\infty}\frac{1}{k}\binom{k+t-2}{t-1}\rho^{k-1}\Bigg]\\
&=(C-D)\,|\tau|\,(1-\rho)^t\Bigg\{(1-B)\left(\frac{1}{(1-\rho)^t}-1\right)\\
&\qquad-\frac{(1-B)-(A-B)cos\lambda}{\rho(t-1)}\left[\sum_{k=0}^{\infty}\binom{k+t-2}{t-2}\rho^k-1-(t-1)\rho\right]\Bigg\}\\
&=(C-D)|\tau|\Bigg\{(1-B)\ \left(1-(1-\rho)^t\right)\\
&\qquad-\frac{(1-B)-(A-B)cos\lambda}{\rho(t-1)}\left[(1-\rho)-(1-\rho)^t-\rho(t-1)(1-\rho)^t\right]\Bigg\}.
\end{aligned}$$

But the last expression is upper bounded by $(A-B)cos\lambda$ if (12) holds, which completes our proof.

Applying Lemma 2 and using the same technique as in the proof of Theorem 6, we have the following result:

Theorem 7. *If* $f\in\mathcal{R}^{\tau}(C,D)$, *then* $\mathcal{I}_{\rho}^{t}f\in\mathcal{K}^{\lambda}(A,B)$ *if*

$$(C-D)\,|\tau|\left[\frac{(1-B)\rho t}{1-\rho}+(A-B)cos\lambda\left(1-(1-\rho)^t\right)\right]\leq(A-B)cos\lambda. \tag{13}$$

Proof. According to Lemma 2 it is sufficient to show that

$$\sum_{k=2}^{\infty} k\left[(k-1)(1-B)+(A-B)cos\lambda\right]|a_k| \leq (A-B)cos\lambda. \tag{14}$$

Since $f \in \mathcal{R}^{\tau}(C,D)$, using Lemma 5 we have

$$|a_k| \leq \frac{(C-D)\,|\tau|}{k},\ k \in \mathbb{N}\setminus\{1\},$$

hence

$$\sum_{k=2}^{\infty} k\left[(k-1)(1-B)+(A-B)cos\lambda\right]\binom{k+t-2}{t-1}\rho^{k-1}(1-\rho)^t\,|a_k|$$
$$\leq \sum_{k=2}^{\infty} k\left[(k-1)(1-B)+(A-B)cos\lambda\right]\binom{k+t-2}{t-1}\rho^{k-1}(1-\rho)^t\frac{(C-D)\,|\tau|}{k}$$
$$=(C-D)\,|\tau|\sum_{k=2}^{\infty}\left[(k-1)(1-B)+(A-B)cos\lambda\right]\binom{k+t-2}{t-1}\rho^{k-1}(1-\rho)^t$$
$$=(C-D)\,|\tau|\left[\frac{(1-B)\rho t}{1-\rho}+(A-B)cos\lambda\left(1-(1-\rho)^t\right)\right].$$

But this last expression is upper bounded by $(A-B)cos\lambda$ if (13) holds, which completes our proof.

4. Properties of a special function

Theorem 8. *If the function $\mathcal{G}_\rho^t$ is given by*

$$\mathcal{G}_\rho^t(z) := \int_0^z \frac{\Phi_\rho^t(\eta)}{\eta}d\eta,\ z\in\mathbb{U}, \tag{15}$$

then $\mathcal{G}_\rho^t \in \mathcal{K}^{\lambda}(A,B)$, if (8) holds.

Proof. According to (6) it follows that

$$\mathcal{G}_\rho^t(z) = z + \sum_{k=2}^{\infty}\binom{k+t-2}{t-1}\rho^{k-1}(1-\rho)^t\frac{z^k}{k},\ z\in\mathbb{U},$$

and using Lemma 2, by a similar proof like those of Theorem 3 we get that $\mathcal{G}_\rho^t \in \mathcal{K}^{\lambda}(A,B)$ if and only if (8) holds.

Theorem 9. *If the function* $\mathcal{G}_\rho^t$ *is given by* (15), *then* $\mathcal{G}_\rho^t \in \mathcal{S}^\lambda(A,B)$ *if*

$$\begin{aligned}&(1-B)\left(1-(1-\rho)^t\right)\\ &-\frac{(1-B)-(A-B)cos\lambda}{\rho(t-1)}\left[(1-\rho)-(1-\rho)^t-\rho(t-1)(1-\rho)^t\right]\\ &\leq (A-B)cos\lambda.\end{aligned} \tag{16}$$

Proof. By Lemma 1, the function

$$\mathcal{G}_\rho^t(z) = z + \sum_{k=2}^{\infty}\binom{k+t-2}{t-1}\rho^{k-1}(1-\rho)^t\frac{z^k}{k}$$

belongs to $\mathcal{S}^\lambda(A,B)$ if the condition (7) is satisfied.

Using similar computations like in the proof of Theorem 6, we get

$$\begin{aligned}&\sum_{k=2}^{\infty}\frac{1}{k}\left[(k-1)(1-B)+(A-B)cos\lambda\right]\binom{k+t-2}{t-1}\rho^{k-1}(1-\rho)^t\\ &=(1-\rho)^t\Bigg[(1-B)\sum_{k=2}^{\infty}\binom{k+t-2}{t-1}\rho^{k-1}\\ &\qquad -[(1-B)-(A-B)cos\lambda]\sum_{k=2}^{\infty}\frac{1}{k}\binom{k+t-2}{t-1}\rho^{k-1}\Bigg]\\ &=(1-\rho)^t\Bigg\{(1-B)\left(\frac{1}{(1-\rho)^t}-1\right)\\ &\qquad -\frac{(1-B)-(A-B)cos\lambda}{\rho(t-1)}\left[\sum_{k=0}^{\infty}\binom{k+t-2}{t-2}\rho^k-1-(t-1)\rho\right]\Bigg\}\\ &=(1-B)\left(1-(1-\rho)^t\right)\\ &\qquad -\frac{(1-B)-(A-B)cos\lambda}{\rho(k-1)}\left[(1-\rho)-(1-\rho)^t-\rho(t-1)(1-\rho)^t\right].\end{aligned}$$

But the last expression is upper bounded by $(A-B)cos\lambda$ It follows that (7) is satisfied if and only if the assumption (16), which proves the result.

Concluding Remarks. Specializing the parameter $\lambda = 0$ we can state various interesting inclusion results (as established in above theorems) for the subclasses $\mathcal{S}[A,B]$ and $\mathcal{K}[A,B]$ (see[11, 12]).

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