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A CERTAIN SUBCLASS OF LOG-HARMONIC MAPPINGS

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Abstract. Let $H(\mathbb{D})$ be the linear space of all analytic functions defined on the open unit disc $\mathbb{D} = \{z : |z| < 1\}$, and $\mathcal{B}$ denote the set of all functions $w \in H(\mathbb{D})$ satisfying $|w(z)| < 1$ for all $z \in \mathbb{D}$. A log-harmonic mapping is a solution of the non-linear elliptic partial differential equation $\overline{f}_{\overline{z}} = w(\frac{\overline{f}}{f})f_z$, where the second dilatation function w belongs to $\mathcal{B}$. In the present paper, we investigate the set of all log-harmonic mappings f defined on $\mathbb{D}$ which are of the form $f(z) = zh(z)\overline{g(z)}$, where h and g are in $H(\mathbb{D})$, $h(0) = g(0) = 1$ and $Re(h(z)/g(z)) > 0$. The class of such functions is denoted by $\mathcal{S}_{LH}(\mathcal{P})$.

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1. Introduction

Let $H(\mathbb{D})$ be the linear space of all analytic functions defined on the open unit disc $\mathbb{D}$, and let $\mathcal{B}$ be the set of all analytic functions $w \in H(\mathbb{D})$ such that $|w(z)| < 1$ for all $z \in \mathbb{D}$. A log-harmonic mapping defined on $\mathbb{D}$ is the solution of the non-linear elliptic partial differential equation

$$\frac{\overline{f}_{\overline{z}}}{\overline{f}} = w\left(\frac{f_z}{f}\right), \tag{1}$$

where the second dilatation function $w \in \mathcal{B}$. The Jacobian

$$J_f = (1 - |w|^2)|f_z|^2$$

is positive and hence, all non-constant log-harmonic mappings are sense-preserving on $\mathbb{D}$. It has been shown that if f is a non-vanishing log-harmonic mappings, then f can be expressed as

$$f(z) = h(z)\overline{g(z)},$$

where h and g are analytic in $\mathbb{D}$, i.e, $h, g \in H(\mathbb{D})$. On the other hand, if f is a non-constant log-harmonic mapping on $\mathbb{D}$ and vanishes at $z = 0$ but is not identically zero, then f admits the representation given by

$$f(z) = z|z|^{2\beta} h(z)\overline{g(z)},$$

where $Re(\beta) > -1/2$ and h and g are analytic functions on $\mathbb{D}$ with $g(0) = 1$ and $h(0) \neq 0$. Univalent log-harmonic mappings have been studied extensively in [1, 2, 3, 4, 5, 8].

Let Ω be the family of functions ϕ which are analytic on $\mathbb{D}$, and satisfy the conditions $\phi(0) = 0, |\phi(z)| < 1$ for all $z \in \mathbb{D}$. If f_1 and f_2 are analytic functions on $\mathbb{D}$, then we say that f_1 is subordinate to f_2 written as $f_1 \prec f_2$, if there exists a Schwarz function $\phi \in \Omega$ such that $f_1(z) = f_2(\phi(z))$. Denote by $\mathcal{P}$ the family of functions p of the form $p(z) = 1 + p_1 z + p_2 z^2 + p_3 z^3 + \cdots$, analytic on $\mathbb{D}$ with $p(0) = 1$ and $Re(p(z)) > 0$ such that p is in $\mathcal{P}$ if and only if

$$p(z) \prec \frac{1+z}{1-z} \Leftrightarrow p(z) = \frac{1+\phi(z)}{1-\phi(z)} \tag{2}$$

for some function $\phi \in \Omega$ and for all $z \in \mathbb{D}$ (see [7]).

Lemma 1. *[7, Caratheodory's lemma] If $p \in \mathcal{P}$ and $p(z) = 1 + \sum_{n=1}^{\infty} p_n z^n$, then $|p_n| \leq 2$ for $n \geq 1$. This inequality is sharp for each n.*

Let f be the set of all log-harmonic mappings f defined on $\mathbb{D}$ which are of the form

$$f(z) = zh(z)\overline{g(z)} \tag{3}$$

where $h(0) = g(0) = 1$ and $Re(h(z)/g(z)) > 0$. The class of such functions is denoted by $\mathcal{S}_{LH}(\mathcal{P})$. In this paper, we will investigate properties of the class $\mathcal{S}_{LH}(\mathcal{P})$.

2. Main Results

Theorem 2. *(Main Characterization) Let $f(z) = zh(z)\overline{g(z)}$ be an element of $\mathcal{S}_{LH}(\mathcal{P})$, then*

$$Re\left(\frac{f(z)}{z}\right) > 0 \Leftrightarrow Re\left(\frac{h(z)}{g(z)}\right) > 0. \tag{4}$$

Proof. Let

$$Re\left(\frac{f(z)}{z}\right) > 0 \Rightarrow Re\left(\frac{zh(z)\overline{g(z)}}{z}\right) = |g(z)|^2 Re\left(\frac{h(z)}{g(z)}\right) > 0.$$

This shows that $Re\left(\frac{h(z)}{g(z)}\right) > 0$. Conversely, suppose $Re\left(\frac{h(z)}{g(z)}\right) > 0$. Then

$$Re\left(\frac{h(z)}{g(z)}\right) > 0 \Rightarrow |g(z)|^2 Re\left(\frac{h(z)}{g(z)}\right) > 0 \Rightarrow Re\left(\frac{zh(z)\overline{g(z)}}{z}\right) = Re\left(\frac{f(z)}{z}\right) > 0.$$

Therefore $f(z) = zh(z)\overline{g(z)} \in \mathcal{S}_{LH}(\mathcal{P})$ satisfies (4).

Theorem 3. *Let* $f(z) = zh(z)\overline{g(z)}$ *be an element of* $\mathcal{S}_{LH}(\mathcal{P})$, *then*

$$e^{1-r}\frac{1}{1-r}\left(\frac{1+r}{1-r}\right)^{\frac{1}{2}} \leq |h(z)| \leq e^{1+r}\frac{1}{1+r}\left(\frac{1+r}{1-r}\right)^{\frac{1}{2}}, \tag{5}$$

$$e^{1-r}\frac{1}{1+r}\left(\frac{1+r}{1-r}\right)^{\frac{1}{2}} \leq |g(z)| \leq e^{1+r}\frac{1}{1-r}\left(\frac{1+r}{1-r}\right)^{\frac{1}{2}}. \tag{6}$$

These inequalities are sharp.

Proof. Since $f(z) = zh(z)\overline{g(z)}$ is an element of $\mathcal{S}_{LH}(\mathcal{P})$, then

$$w(z) = \frac{\overline{f_{\overline{z}}}}{\overline{f}}.\frac{f}{f_z} = \frac{z\frac{g'(z)}{g(z)}}{1+z\frac{h'(z)}{h(z)}}, \tag{7}$$

where

$$\frac{zf_z}{f} = 1 + z\frac{h'(z)}{h(z)} \quad \text{and} \quad \frac{\overline{z}\overline{f}_{\overline{z}}}{\overline{f}} = z\frac{g'(z)}{g(z)}.$$

The equality (7) shows that the second dilatation of f satisfies the conditions of Schwarz lemma. Using the definition of $\mathcal{S}_{LH}(\mathcal{P})$, the equation in (7) and the definition of subordination, then we obtain

$$\frac{1+z\frac{h'(z)}{h(z)}}{1-z\frac{p'(z)}{p(z)}} = \frac{1}{1-w(z)} \Leftrightarrow \frac{1+z\frac{h'(z)}{h(z)}}{1-z\frac{p'(z)}{p(z)}} \prec \frac{1}{1-z}.$$

On the other hand, the transformation $(\frac{1}{1-z})$ maps $|z| = r$ onto the disc with the centre $C(r) = 1/(1-r^2)$ and the radius $\rho(r) = r/(1-r^2)$, then we have

$$\left|\frac{1+z\frac{h'(z)}{h(z)}}{1-z\frac{p'(z)}{p(z)}} - \frac{1}{1-r^2}\right| \leq \frac{r}{1-r^2}. \tag{8}$$

Simple calculations in (8) gives

$$\frac{1}{1+r}\left|1-z\frac{p'(z)}{p(z)}\right| \leq \left|1+z\frac{h'(z)}{h(z)}\right| \leq \frac{1}{1-r}\left|1-z\frac{p'(z)}{p(z)}\right|. \tag{9}$$

Since $p \in \mathcal{P}$, then we have

$$-\left|1-z\frac{p'(z)}{p(z)}\right| \geq -\left(1+\frac{2r}{1-r^2}\right), \tag{10}$$

and

$$\left|1-z\frac{p'(z)}{p(z)}\right| \leq 1+\frac{2r}{1-r^2}. \tag{11}$$

Considering inequalities in (9), (10) and (11), we obtain

$$-\frac{1}{1+r}\left(1+\frac{2r}{1-r^2}\right) \leq \left|1+z\frac{h'(z)}{h(z)}\right| \leq \frac{1}{1-r}\left(1+\frac{2r}{1-r^2}\right). \tag{12}$$

On the other hand, we have

$$Re\left(1+z\frac{h'(z)}{h(z)}\right) = 1+r\frac{\partial}{\partial r}\log|h(z)|.$$

Therefore, the inequality in (12) can be written by

$$-\frac{1}{r(1-r)}\left(1+\frac{2r}{1-r^2}\right)-\frac{1}{r} \leq \frac{\partial}{\partial r}\log|h(z)| \leq \frac{1}{r(1+r)}\left(1+\frac{2r}{1-r^2}\right)-\frac{1}{r}. \tag{13}$$

Integrating both sides of (13) from 0 to r, we obtain (5). Since $p(z) = \dfrac{h(z)}{g(z)}$, if we use the growth theorem for the class $\mathcal{P}$ given by

$$\frac{1-r}{1+r} \leq |p(z)| \leq \frac{1+r}{1-r},$$

in (5), then we obtain (6). These inequalities are sharp, because

$$w(z) = \frac{z\dfrac{g'(z)}{g(z)}}{1+z\dfrac{h'(z)}{h(z)}} \Rightarrow \frac{1-z\dfrac{p'(z)}{p(z)}}{1+z\dfrac{h'(z)}{h(z)}} = 1-w(z). \tag{14}$$

If we take $p(z) = \dfrac{1+z}{1-z}, w(z) = z$, then the inequality in (14) can be written by

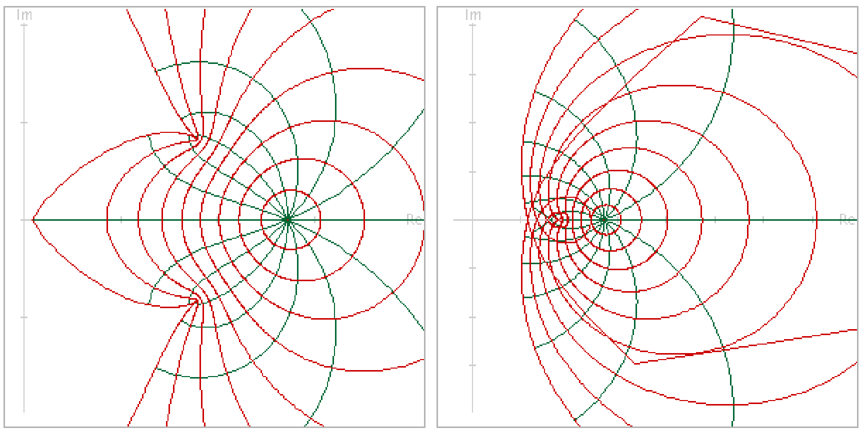
$$\frac{h'(z)}{h(z)} = \frac{3-z^2}{(1+z)(1-z)^2}. \tag{15}$$

Then, respectively, we obtain

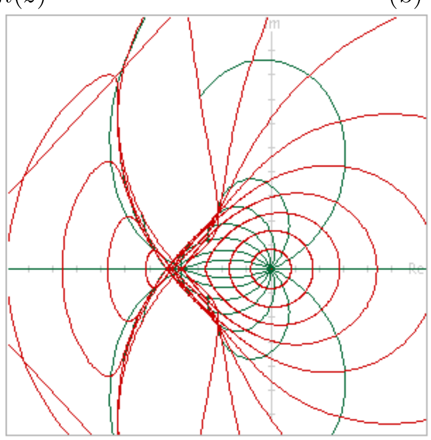
$$h(z) = e^{1-z}.\frac{(1+z)^{\frac{1}{2}}}{(1-z)^{\frac{3}{2}}},$$

and

$$g(z) = \frac{h(z)}{p(z)} = e^{1-z}.\frac{(1-z)^{-\frac{1}{2}}}{(1+z)^{\frac{1}{2}}}.$$



(a) $h(z)$ (b) $g(z)$



(c) $f(z)$

Figure 1: $h(z)$, $g(z)$ and $f(z) = zh(z)\overline{g(z)}$

Theorem 4. *Let $f(z) = zh(z)\overline{g(z)}$ be an element of $\mathcal{S}_{LH}(\mathcal{P})$, then*

$$F_2(r) \leq |h'(z)| \leq F_1(r), \tag{16}$$

where

$$F_1(r) = e^{1+r}\frac{1}{1+r}\left(\frac{1+r}{1-r}\right)^{\frac{1}{2}}\left[\frac{1}{1-r}\left(1+\frac{2r}{1-r^2}\right)+\frac{2}{1-r^2}\right],$$

$$F_2(r) = e^{1-r}\frac{1}{1-r}\left(\frac{1+r}{1-r}\right)^{\frac{1}{2}}\left[\frac{1}{1+r}\left(1+\frac{2r}{1-r^2}\right)-\frac{2}{1-r^2}\right],$$

and

$$G(-r)\left(\frac{1+r}{1-r}\right)^{\frac{1}{2}}\left(1+\frac{2r}{1-r^2}\right) \leq |g'(z)| \leq G(r)\left(\frac{1+r}{1-r}\right)^{\frac{1}{2}}\left(1+\frac{2r}{1-r^2}\right), \tag{17}$$

where

$$G(r) = e^{1+r}\frac{1}{(1-r)^2}.$$

These inequalities are sharp.

Proof. Since the second dilatation of f satisfies the conditions Schwarz lemma, then we can write

$$-r \leq \left|\frac{z\frac{g'(z)}{g(z)}}{1+z\frac{h'(z)}{h(z)}}\right| \leq r \Leftrightarrow$$

$$-r\left|1+z\frac{h'(z)}{h(z)}\right| \leq \left|z\frac{g'(z)}{g(z)}\right| \leq r\left|1+z\frac{h'(z)}{h(z)}\right| \tag{18}$$

Using the inequality (12) in (18), we get (17). Since $p(z) = \dfrac{h(z)}{g(z)}$, then we obtain

$$\frac{zp'(z)}{p(z)} = \frac{zh'(z)}{h(z)} - \frac{zg'(z)}{g(z)}$$

Simple calculations shows that the above equality gives (16).

Corollary 5. *Let $f(z) = zh(z)\overline{g(z)}$ be an element of $\mathcal{S}_{LH}(\mathcal{P})$, then*

$$\frac{r}{(1-r)^2}e^{2(1-r)} \leq |f| \leq \frac{r}{(1-r)^2}e^{2(1+r)} \tag{19}$$

This inequality is sharp for the extremal function given in Theorem 3.

Proof. Since $f(z) = zh(z)\overline{g(z)}$, then taking modulus on both sides we get

$$|f| = |zh(z)\overline{g(z)}| = |z||h(z)||g(z)|.$$

Using inequalities (5) and (6), we get (19).

Corollary 6. *Let $f(z) = zh(z)\overline{g(z)}$ be an element of $\mathcal{S}_{LH}(\mathcal{P})$, then*

$$\frac{1}{1+r}\frac{1}{(1-r)^3}\left(1+\frac{2r}{1-r^2}\right)^2 e^{4(1-r)} \le J_f \le \frac{1}{(1-r)^6}\left(1+\frac{2r}{1-r^2}\right)e^{4(1+r)} \qquad (20)$$

This inequality is sharp for the extremal function given in Theorem 3.

Proof. Using the definition of Jacobian of f, then we have

$$|f_z|^2(1-r^2) \le J_f = (1-|w(z)|^2)|f_z|^2 \le |f_z|^2.$$

Also we have $\frac{zf_z}{f} = 1 + z\frac{h'(z)}{h(z)}$. Thus, considering Corollary 5 and inequality in (12), we obtain (20).

Theorem 7. *The radius of starlikenes of the class $\mathcal{S}_{LH}(\mathcal{P})$ is the smallest positive root of the equation $\varphi(r) = 1 - 2r - r^2$ in $(0,1)$.*

Proof. The radius of starlikenes of the class sense-preserving log-harmonic mappings is defined by

$$r_s = \sup\left\{r : Re\frac{zf_z - \overline{z}f_{\overline{z}}}{f} > 0,\ 0 < r < 1\right\}.$$

Due to the definition of the class $\mathcal{S}_{LH}(\mathcal{P})$,

$$Re\frac{h(z)}{g(z)} > 0 \Rightarrow \frac{h(z)}{g(z)} = p(z) \Rightarrow$$

$$\frac{zf_z - \overline{z}f_{\overline{z}}}{f} = 1 + z\frac{h'(z)}{h(z)} - \overline{z}\frac{\overline{g'(z)}}{\overline{g(z)}}$$

$$\begin{aligned} Re\frac{zf_z - \overline{z}f_{\overline{z}}}{f} &= Re\left(1 + z\frac{h'(z)}{h(z)} - z\frac{g'(z)}{g(z)}\right) \\ &= Re\left(1 + z\frac{p'(z)}{p(z)}\right) \\ &\ge 1 - \frac{2r}{1-r^2} = \frac{1-2r-r^2}{1-r^2} \end{aligned}$$

Therefore, $\varphi(r) = 1 - 2r - r^2 \Rightarrow \varphi(1) = -2 < 0$, $\varphi(0) = 1$. Thus, the smallest positive root r_0 of the equation $\varphi(r) = 1 - 2r - r^2 = 0$ lies on 0 and 1. Thus $Re\dfrac{zf_z - \overline{z}f_{\overline{z}}}{f} > 0$ is valid for $|z| = r < r_0$. Hence the radius of starlikeness r_s for $\mathcal{S}_{LH}(\mathcal{P})$ is not less than r_0.

Theorem 8. *Let $f(z) = zh(z)\overline{g(z)}$ be an element $\mathcal{S}_{LH}(\mathcal{P})$, where $h(z) = 1 + a_1z + a_2z^2...$ and $g(z) = 1 + b_1z + b_2z^2...$, then we have*

$$(i) \quad |a_n| \leq 2\sum_{k=0}^{n-1}|b_k| + |b_n|, \quad |b_0| = 1 \tag{21}$$

$$(ii) \quad |a_{n+1} - b_{n+1}| \leq 4 + 4\sum_{k=1}^{n} Re a_k\overline{b}_k, \tag{22}$$

$$(iii) \quad \sum_{k=1}^{n}|a_k - b_k|^2 \leq 4 + \sum_{k=1}^{n-1}|a_k^2 + b_k^2| \tag{23}$$

These inequalities are sharp.

Proof. (i) Since

$$Re\frac{h(z)}{g(z)} > 0 \Rightarrow \frac{h(z)}{g(z)} = p(z) \Rightarrow h(z) = g(z)p(z),$$

then we write

$$(1+a_1z+a_2z^2+...+a_nz^n+...) = (1+b_1z+b_2z^2+...+b_nz^n+...)(1+p_1z+p_2z^2+...+p_nz^n+...).$$

Comparing the coefficients of z^n on both sides, we get

$$a_n = b_1p_{n-1} + b_2p_{n-3} + ... + p_1b_{n-1} + b_n.$$

In view of Lemma 1, we obtain the coefficient inequality given in (21).

(ii) Since $p \in \mathcal{P}$, then the conditions $p(0) = 1$ and $Re(p(z)) > 0$ are satisfied. From subordination condition given in (2), we obtain

$$p(z) = \frac{1 + \phi(z)}{1 - \phi(z)} \Leftrightarrow \phi(z) = \frac{p(z) - 1}{p(z) + 1}$$

Therefore

$$\phi(z) = \frac{p(z) - 1}{p(z) + 1} = \frac{\frac{h(z)}{g(z)} - 1}{\frac{h(z)}{g(z)} + 1} = \frac{h(z) - g(z)}{h(z) + g(z)}$$

which gives

$$(h(z) - g(z)) = \phi(z)(h(z) + g(z)).$$

This shows that $(h(z) - g(z))$ is majorized by $(h(z) + g(z))$. Using the coefficient inequality for majorized functions, we write

$$|a_{n+1} - b_{n+1}|^2 = \sum_{k=0}^{n} |a_{k+1} - b_{k+1}|^2 \leq 4 + \sum_{k=0}^{n} |a_k + b_k|^2$$

$$|a_{n+1} - b_{n+1}|^2 = \sum_{k=0}^{n} (a_{k+1} - b_{k+1})(\overline{a}_{k+1} - \overline{b}_{k+1}) \leq 4 + \sum_{k=0}^{n} (a_k + b_k)(\overline{a}_k - \overline{b}_k)$$

which gives (22). This method is based on the Rogogonski method [9].

(iii) Using the equality $(h(z) - g(z)) = \phi(z)(h(z) + g(z))$ and Clunie method [6], we write

$$\sum_{k=1}^{n} (a_k - b_k) z^k + \sum_{k=n+1}^{\infty} (a_k - b_k) z^k = \left[4 + \sum_{k=1}^{n-1} (a_k + b_k) z^k + \sum_{k=n}^{\infty} (a_k + b_k) z^k\right] \left(\sum_{k=1}^{\infty} c_k z^k\right) \Rightarrow$$

$$\sum_{k=1}^{n} |a_k - b_k|^2 r^{2k} + \sum_{k=n+1}^{\infty} |a_k|^2 r^{2k} \leq \left[4 + \sum_{k=1}^{n-1} |a_k + b_k|^2 r^{2k}\right]$$

Letting $r \to 1$, we obtain desired bound given in (23).

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