Acta Mathematica Academiae Paedagogicae Nyíregyháziensis
34 (2023), 78–95.
www.emis.de/journals
ISSN 1786-0091

HERMITE–HADAMARD TYPE INEQUALITIES FOR TWICE DIFFERENTIABLE GENERALIZED BETA-PREINVEX FUNCTIONS VIA k-FRACTIONAL INTEGRALS

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Abstract. In the present paper, a new class of generalized beta-preinvex function is introduced and some new integral inequalities for the left hand side of the Gauss–Jacobi type quadrature formula involving generalized beta-preinvex functions are given. Moreover, some Hermite–Hadamard type inequalities for generalized beta-preinvex functions that are twice differentiable via k-fractional integrals are established. At the end, some applications to special means are given. These general inequalities give us some new estimates for Hermite–Hadamard type k-fractional integral inequalities.

1. Introduction

The following notation is used throughout this paper. We use I to denote an interval on the real line $\mathbb{R} = (-\infty, +\infty)$ and I° to denote the interior of I. For any subset $K \subseteq \mathbb{R}^n, K^\circ$ is used to denote the interior of K. The symbol $\mathbb{R}^n$ is used to denote a generic n-dimensional vector space. The nonnegative real numbers are denoted by $\mathbb{R}_\circ = [0, +\infty)$. The set of integrable functions on the interval $[a, b]$ is denoted by $L_1[a, b]$.

The following inequality, named Hermite–Hadamard inequality, is one of the most famous inequalities in the literature for convex functions.

Theorem 1. *Let $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}$ be a convex function on an interval I of real numbers and $a, b \in I$ with $a < b$. Then the following inequality holds:*

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a}\int_a^b f(x)dx \leq \frac{f(a)+f(b)}{2}.$$

The following definition will be used in the sequel.

2020 *Mathematics Subject Classification.* 26A33, 26A51, 33B15, 26B25, 26D07, 26D10, 26D15.

Key words and phrases. Hermite–Hadamard type inequality, Hölder's inequality, power mean inequality, Riemann–Liouville fractional integral, k-fractional integral, s-convex function in the second sense, m-invex, P-function, special means.

Definition 1. The hypergeometric function ${}_2F_1(a,b;c;z)$ is defined by

$${}_2F_1(a,b;c;z) = \frac{1}{\beta(b,c-b)} \int_0^1 t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a}dt$$

for $c > b > 0$ and $|z| < 1$, where $\beta(x,y)$ is the Euler beta function for all $x, y > 0$.

In recent years, various generalizations, extensions and variants of such inequalities have been obtained, see [1, 2]. For other recent results concerning Hermite–Hadamard type inequalities through various classes of convex functions, see [3, 4, 5] and the references cited therein.

Fractional calculus, see [4] and the references cited therein, was introduced at the end of the nineteenth century by Liouville and Riemann, the subject of which has become a rapidly growing area and has found applications in diverse fields ranging from physical sciences and engineering to biological sciences and economics.

Definition 2. Let $f \in L_1[a,b]$. The Riemann–Liouville integrals $J^{\alpha}_{a+}f$ and $J^{\alpha}_{b-}f$ of order $\alpha > 0$ with $a \geq 0$ are defined by

$$J^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)dt, \quad x > a,$$

and

$$J^{\alpha}_{b-}f(x) = \frac{1}{\Gamma(\alpha)} \int_x^b (t-x)^{\alpha-1} f(t)dt, \quad b > x,$$

where $\Gamma(\alpha) = \displaystyle\int_0^{+\infty} e^{-u}u^{\alpha-1}du$.

Here $J^0_{a+}f(x) = J^0_{b-}f(x) = f(x)$. In the case of $\alpha = 1$, the fractional integral reduces to the classical integral.

Due to the wide application of fractional integrals, some authors extended to study fractional Hermite–Hadamard type inequalities for functions of different classes, see [4, 6] and the references cited therein.

Definition 3. ([7]) If $k > 0$, then k-Gamma function Γ_k is defined as

$$\Gamma_k(\alpha) = \lim_{n\to\infty} \frac{n!k^n(nk)^{\frac{\alpha}{k}-1}}{(\alpha)_{n,k}}.$$

If $\mathrm{Re}(\alpha) > 0$ then k-Gamma function in integral form is defined as

$$\Gamma_k(\alpha) = \int_0^{\infty} t^{\alpha-1}e^{-\frac{t^k}{k}}dt$$

with the following property

$$\Gamma_k(\alpha + k) = \alpha\Gamma_k(\alpha).$$

Definition 4. ([8]) Let $f \in L_1[a,b]$. Then k-fractional integrals of order $\alpha, k > 0$ with $a \geq 0$ are defined as

$$I_{a+}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_a^x (x-t)^{\frac{\alpha}{k}-1} f(t)dt, \quad x > a,$$

and

$$I_{b-}^{\alpha,k} f(x) = \frac{1}{k\Gamma_k(\alpha)} \int_x^b (t-x)^{\frac{\alpha}{k}-1} f(t)dt, \quad b > x.$$

For $k = 1$, k-fractional integrals give Riemann–Liouville integrals.

Now, let us recall some definitions of various convex functions.

Definition 5. ([9]) A non-negative function $f : I \subseteq \mathbb{R} \longrightarrow \mathbb{R}_\circ$ is said to be P-function (or P-convex), if

$$f(tx + (1-t)y) \leq f(x) + f(y)$$

for all $x, y \in I,\ t \in [0,1]$.

Definition 6. ([10]) A function $f : \mathbb{R}_\circ \longrightarrow \mathbb{R}$ is said to be s-convex in the second sense, if

$$f(tx + (1-t)y) \leq t^s f(x) + (1-t)^s f(y)$$

for all $x, y \in \mathbb{R}_\circ,\ t \in [0,1]$ and $s \in (0,1]$.

It is clear that a 1-convex function must be convex on $\mathbb{R}_\circ$ as usual. The s-convex functions in the second sense have been investigated in [10].

Definition 7. ([11]) A set $K \subseteq \mathbb{R}^n$ is said to be invex with respect to the mapping $\sigma : K \times K \longrightarrow \mathbb{R}^n$, if $x + t\sigma(y,x) \in K$ for every $x, y \in K$ and $t \in [0,1]$.

Notice that every convex set is invex with respect to the mapping $\sigma(y,x) = y - x$, but the converse is not necessarily true. For more details, see [11, 12] and the references therein.

Definition 8. ([13]) The function f defined on the invex set $K \subseteq \mathbb{R}^n$ is said to be preinvex with respect to σ, if for every $x, y \in K$ and $t \in [0,1]$, we have that

$$f(x + t\sigma(y,x)) \leq (1-t)f(x) + tf(y).$$

The concept of preinvexity is more general than convexity since every convex function is preinvex with respect to the mapping $\sigma(y,x) = y - x$, but the converse is not true.

The Gauss–Jacobi type quadrature formula has the following

$$\int_a^b (x-a)^p (b-x)^q f(x)dx = \sum_{k=0}^{+\infty} B_{m,k} f(\gamma_k) + R_m^\star |f|, \tag{1}$$

for certain $B_{m,k}, \gamma_k$ and rest $R_m^\star |f|$.

Recently, Liu in [14] obtained several integral inequalities for the left hand side of (1) under the definition of P-function (Definition 5). Also in [15], Özdemir et al. established several integral inequalities concerning the left-hand side of (1) via some kinds of convexity.

Motivated by these results, in Section 2, the notion of generalized beta-preinvex function is introduced and some new integral inequalities for the left hand side of (1) involving generalized beta-preinvex functions are given. In Section 3, some Hermite–Hadamard type inequalities for generalized beta-preinvex functions that are twice differentiable via k-fractional integrals are established. In Section 4, some applications to special means are obtained. In Section 5, some conclusions and future research are given. These general inequalities give us some new estimates for Hermite–Hadamard type k-fractional integral inequalities.

2. New integral inequalities

Definition 9. ([16]) A set $K \subseteq \mathbb{R}^n$ is said to be m-invex with respect to the mapping $\sigma : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ for some fixed $m \in (0,1]$, if $mx + t\sigma(y,x,m) \in K$ holds for each $x, y \in K$ and any $t \in [0,1]$.

Remark 1. In Definition 9, under certain conditions, the mapping $\sigma(y,x,m)$ could reduce to $\sigma(y,x)$.

Definition 10. ([17]) Let $K \subseteq \mathbb{R}^n$ be an open m-invex set with respect to $\sigma : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ and a continuous function $\phi : I \longrightarrow K$. For $f : K \longrightarrow \mathbb{R}$ and any fixed $s, m \in (0,1]$, if

$$f\left(m\phi(x) + \lambda\sigma(\phi(y), \phi(x), m)\right) \leq m(1-\lambda)^s f(\phi(x)) + \lambda^s f(\phi(y))$$

is valid for all $x, y \in I, \lambda \in [0,1]$, then we say that $f(x)$ is a generalized (s,m,ϕ)-preinvex function with respect to σ.

Next we give a new definition, to be referred as generalized beta-preinvex function.

Definition 11. Let $K \subseteq \mathbb{R}^n$ be an open m-invex set with respect to $\sigma : K \times K \times (0,1] \longrightarrow \mathbb{R}^n$ and a continuous function $\phi : I \longrightarrow K$. For $f : K \longrightarrow \mathbb{R}$ and some fixed $m \in (0,1]$, if

$$f\left(m\phi(x) + t\sigma(\phi(y), \phi(x), m)\right) \leq mt^p(1-t)^q f(\phi(x)) + t^q(1-t)^p f(\phi(y))$$

is valid for all $x, y \in I, t \in [0,1]$ where $p, q > -1$, then we say that $f(x)$ is a generalized beta-preinvex function with respect to σ.

Remark 2. In Definition 11, it is worthwhile to note that the class of generalized beta-preinvex functions is a generalization of the class of P-convex functions given in Definition 5, s-convex in the second sense functions given in Definition 6 and generalized (s,m,ϕ)-preinvex functions given in Definition 10, for $(p,q) = \{(0,0),(s,0),(0,s)\}$, where $m = 1$, $\sigma(\phi(y),\phi(x),m) = \phi(y) - m\phi(x)$ and $\phi(x) = x$, $\forall x \in I$.

In this section, in order to prove our main results regarding some new integral inequalities involving generalized beta-preinvex functions, we need the following lemma.

Lemma 1. *Let $\phi : I \longrightarrow K$ be a continuous function. Assume that $f : K = [m\phi(a), m\phi(a) + \sigma(\phi(b), \phi(a), m)] \longrightarrow \mathbb{R}$ is a continuous function on the interval of real numbers K° with respect to $\sigma : K \times K \times (0,1] \longrightarrow \mathbb{R}$, for $\sigma(\phi(b), \phi(a), m) > 0$. Then for some fixed $m \in (0,1]$ and any fixed $p, q > 0$, we have*

$$\int_{m\phi(a)}^{m\phi(a)+\sigma(\phi(b),\phi(a),m)} (x - m\phi(a))^p (m\phi(a) + \sigma(\phi(b), \phi(a), m) - x)^q f(x) dx$$
$$= \sigma^{p+q+1}(\phi(b), \phi(a), m) \int_0^1 t^p (1-t)^q f(m\phi(a) + t\sigma(\phi(b), \phi(a), m)) dt.$$

Proof. It is easy to observe that

$$\int_{m\phi(a)}^{m\phi(a)+\sigma(\phi(b),\phi(a),m)} (x - m\phi(a))^p (m\phi(a) + \sigma(\phi(b), \phi(a), m) - x)^q f(x) dx$$
$$= \sigma(\phi(b), \phi(a), m) \int_0^1 (m\phi(a) + t\sigma(\phi(b), \phi(a), m) - m\phi(a))^p$$
$$\times (m\phi(a) + \sigma(\phi(b), \phi(a), m) - m\phi(a) - t\sigma(\phi(b), \phi(a), m))^q$$
$$\times f(m\phi(a) + t\sigma(\phi(b), \phi(a), m)) dt$$
$$= \sigma^{p+q+1}(\phi(b), \phi(a), m) \int_0^1 t^p (1-t)^q f(m\phi(a) + t\sigma(\phi(b), \phi(a), m)) dt.$$

So, the proof of this lemma is completed. □

Theorem 2. *Let $\phi : I \longrightarrow K$ be a continuous function. Assume that $f : K = [m\phi(a), m\phi(a) + \sigma(\phi(b), \phi(a), m)] \longrightarrow \mathbb{R}$ is a continuous function on the interval of real numbers K° with $\sigma(\phi(b), \phi(a), m) > 0$. If $k > 1$ and $|f|^{\frac{k}{k-1}}$ is a generalized beta-preinvex function on an open m-invex set K with respect to $\sigma : K \times K \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$, where $r, s > -1$, then for any fixed $p, q > 0$,*

$$\int_{m\phi(a)}^{m\phi(a)+\sigma(\phi(b),\phi(a),m)} (x - m\phi(a))^p (m\phi(a) + \sigma(\phi(b), \phi(a), m) - x)^q f(x) dx$$
$$\leq \sigma^{p+q+1}(\phi(b), \phi(a), m) \Big[\beta(r+1, s+1)\Big]^{\frac{k-1}{k}} \Big[\beta(kp+1, kq+1)\Big]^{\frac{1}{k}}$$
$$\times \Big[m|f(\phi(a))|^{\frac{k}{k-1}} + |f(\phi(b))|^{\frac{k}{k-1}}\Big]^{\frac{k-1}{k}}.$$

Proof. Since $|f|^{\frac{k}{k-1}}$ is a generalized beta-preinvex function on K, combining Lemma 1, Hölder inequality and properties of the modulus, we get

$$\begin{aligned}
&\int_{m\phi(a)}^{m\phi(a)+\sigma(\phi(b),\phi(a),m)} (x-m\phi(a))^p(m\phi(a)+\sigma(\phi(b),\phi(a),m)-x)^q f(x)dx \\
&\leq |\sigma(\phi(b),\phi(a),m)|^{p+q+1}\left[\int_0^1 t^{kp}(1-t)^{kq}dt\right]^{\frac{1}{k}} \\
&\quad\times\left[\int_0^1 \left|f(m\phi(a)+t\sigma(\phi(b),\phi(a),m))\right|^{\frac{k}{k-1}}dt\right]^{\frac{k-1}{k}} \\
&\leq \sigma^{p+q+1}(\phi(b),\phi(a),m)\Big[\beta(kp+1,kq+1)\Big]^{\frac{1}{k}} \\
&\quad\times\left[\int_0^1 \left(mt^r(1-t)^s|f(\phi(a))|^{\frac{k}{k-1}}+t^s(1-t)^r|f(\phi(b))|^{\frac{k}{k-1}}\right)dt\right]^{\frac{k-1}{k}} \\
&= \sigma^{p+q+1}(\phi(b),\phi(a),m)\Big[\beta(r+1,s+1)\Big]^{\frac{k-1}{k}}\Big[\beta(kp+1,kq+1)\Big]^{\frac{1}{k}} \\
&\quad\times\Big[m|f(\phi(a))|^{\frac{k}{k-1}}+|f(\phi(b))|^{\frac{k}{k-1}}\Big]^{\frac{k-1}{k}}.
\end{aligned}$$

So, the proof of this theorem is completed. □

Corollary 1. *Under the conditions of Theorem 2 for $r=0$, we get ([17], Theorem 2.2).*

Theorem 3. *Let $\phi: I \longrightarrow K$ be a continuous function. Assume that $f: K=[m\phi(a), m\phi(a)+\sigma(\phi(b),\phi(a),m)] \longrightarrow \mathbb{R}$ is a continuous function on the interval of real numbers K° with $\sigma(\phi(b),\phi(a),m)>0$. If $l\geq 1$ and $|f|^l$ is a generalized beta-preinvex function on an open m-invex set K with respect to $\sigma: K\times K\times(0,1]\longrightarrow\mathbb{R}$ for some fixed $m\in(0,1]$ where $r,s>-1$, then for any fixed $p,q>0$,*

$$\begin{aligned}
&\int_{m\phi(a)}^{m\phi(a)+\sigma(\phi(b),\phi(a),m)} (x-m\phi(a))^p(m\phi(a)+\sigma(\phi(b),\phi(a),m)-x)^q f(x)dx \\
&\leq \sigma^{p+q+1}(\phi(b),\phi(a),m)\big[\beta(p+1,q+1)\big]^{\frac{l-1}{l}} \\
&\quad\times\big[m\beta\left(p+r+1,q+s+1\right)|f(\phi(a))|^l+\beta\left(p+s+1,q+r+1\right)|f(\phi(b))|^l\big]^{\frac{1}{l}}.
\end{aligned}$$

Proof. Since $|f|^l$ is a generalized beta-preinvex function on K, using Lemma 1, the well-known power mean inequality and properties of the modulus, we get

$$\begin{aligned}
&\int_{m\phi(a)}^{m\phi(a)+\sigma(\phi(b),\phi(a),m)}(x-m\phi(a))^p(m\phi(a)+\sigma(\phi(b),\phi(a),m)-x)^q f(x)dx\\
&=\sigma^{p+q+1}(\phi(b),\phi(a),m)\\
&\quad\times\int_0^1\Big[t^p(1-t)^q\Big]^{\frac{l-1}{l}}\Big[t^p(1-t)^q\Big]^{\frac{1}{l}}f(m\phi(a)+t\sigma(\phi(b),\phi(a),m))dt\\
&\leq|\sigma(\phi(b),\phi(a),m)|^{p+q+1}\left[\int_0^1 t^p(1-t)^q dt\right]^{\frac{l-1}{l}}\\
&\quad\times\left[\int_0^1 t^p(1-t)^q\big|f(m\phi(a)+t\sigma(\phi(b),\phi(a),m))\big|^l dt\right]^{\frac{1}{l}}\\
&\leq\sigma^{p+q+1}(\phi(b),\phi(a),m)\Big[\beta(p+1,q+1)\Big]^{\frac{l-1}{l}}\\
&\quad\times\left[\int_0^1 t^p(1-t)^q\left(mt^r(1-t)^s|f(\phi(a))|^l+t^s(1-t)^r|f(\phi(b))|^l\right)dt\right]^{\frac{1}{l}}\\
&=\sigma^{p+q+1}(\phi(b),\phi(a),m)\Big[\beta(p+1,q+1)\Big]^{\frac{l-1}{l}}\\
&\quad\times\Big[m\beta\left(p+r+1,q+s+1\right)|f(\phi(a))|^l+\beta\left(p+s+1,q+r+1\right)|f(\phi(b))|^l\Big]^{\frac{1}{l}}.
\end{aligned}$$

So, the proof of this theorem is completed. □

Corollary 2. *Under the conditions of Theorem 3 for $r=0$, we get ([17], Theorem 2.3).*

3. Novel Hermite–Hadamard-type inequalities

In this section, in order to prove our main results regarding some generalizations of Hermite–Hadamard type inequalities for generalized beta-preinvex functions via k-fractional integrals, we need the following new integral identity.

Lemma 2. *Let $\phi: I \longrightarrow K$ be a continuous function. Suppose $K=[m\phi(a), m\phi(a)+\sigma(\phi(b),\phi(a),m)]\subseteq\mathbb{R}$ be an open m-invex subset with respect to $\sigma: K\times K\times(0,1]\longrightarrow\mathbb{R}$ for some fixed $m\in(0,1]$ where $r\in[0,1]$ and let $\sigma(\phi(b),\phi(a),m)>0$. Assume that $f: K\longrightarrow\mathbb{R}$ be a twice differentiable function on K° and $f''\in L_1(K)$. Then for $\alpha,k>0$, the following identity for* *k-fractional Riemann–Liouville integrals holds:*

$$\frac{\sigma^{\frac{\alpha}{k}+1}(\phi(x),\phi(a),m)f'\left(m\phi(a)+\frac{\sigma(\phi(x),\phi(a),m)}{r+1}\right)-\sigma^{\frac{\alpha}{k}+1}(\phi(x),\phi(b),m)f'(m\phi(b))}{(r+1)\left(\frac{\alpha}{k}+1\right)\sigma(\phi(b),\phi(a),m)}$$

$$-\frac{\sigma^{\frac{\alpha}{k}}(\phi(x),\phi(a),m)f\left(m\phi(a)+\frac{\sigma(\phi(x),\phi(a),m)}{r+1}\right)+\sigma^{\frac{\alpha}{k}}(\phi(x),\phi(b),m)f(m\phi(b))}{\sigma(\phi(b),\phi(a),m)}$$

$$+\frac{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{\sigma(\phi(b),\phi(a),m)}$$

$$\times\left[I^{\alpha,k}_{(m\phi(b))^+}f\left(m\phi(b)+\frac{\sigma(\phi(x),\phi(b),m)}{r+1}\right)+I^{\alpha,k}_{\left(m\phi(a)+\frac{\sigma(\phi(x),\phi(a),m)}{r+1}\right)^-}f(m\phi(a))\right]$$

$$=\frac{\sigma^{\frac{\alpha}{k}+2}(\phi(x),\phi(a),m)}{(r+1)^2\left(\frac{\alpha}{k}+1\right)\sigma(\phi(b),\phi(a),m)}$$

$$\times\int_0^1 t^{\frac{\alpha}{k}+1}f''\left(m\phi(a)+\frac{t}{r+1}\sigma(\phi(x),\phi(a),m)\right)dt$$

$$-\frac{\sigma^{\frac{\alpha}{k}+2}(\phi(x),\phi(b),m)}{(r+1)^2\left(\frac{\alpha}{k}+1\right)\sigma(\phi(b),\phi(a),m)}$$

$$\times\int_0^1 (1-t)^{\frac{\alpha}{k}+1}f''\left(m\phi(b)+\frac{t}{r+1}\sigma(\phi(x),\phi(b),m)\right)dt.$$

We denote

$$\begin{aligned}
&A_{\alpha,k}(x;\sigma,\phi,w,m,a,b)\\
&\quad=\frac{\sigma^{\frac{\alpha}{k}+2}(\phi(x),\phi(a),m)}{(w+1)^2\left(\frac{\alpha}{k}+1\right)\sigma(\phi(b),\phi(a),m)}\\
&\quad\times\int_0^1 t^{\frac{\alpha}{k}+1}f''\left(m\phi(a)+\frac{t}{w+1}\sigma(\phi(x),\phi(a),m)\right)dt\\
&\quad-\frac{\sigma^{\frac{\alpha}{k}+2}(\phi(x),\phi(b),m)}{(w+1)^2\left(\frac{\alpha}{k}+1\right)\sigma(\phi(b),\phi(a),m)}\\
&\quad\times\int_0^1 (1-t)^{\frac{\alpha}{k}+1}f''\left(m\phi(b)+\frac{t}{w+1}\sigma(\phi(x),\phi(b),m)\right)dt.
\end{aligned}\tag{2}$$

Proof. A simple proof of the equality can be done by performing two integration by parts in the integrals from the right side of relation (2) and changing the variables. The details are left to the interested reader. □

Using relation (2), the following results can be obtained for the corresponding version for power of the absolute value of the second derivative.

Theorem 4. *Let $\phi : I \longrightarrow A$ be a continuous function. Suppose that* $A = [m\phi(a), m\phi(a)+\sigma(\phi(b),\phi(a),m)] \subseteq \mathbb{R}$ *is an open m-invex subset with respect* *to* $\sigma : A \times A \times (0,1] \longrightarrow \mathbb{R}$ *for some fixed* $m \in (0,1]$ *and* $\forall w \in [0,1]$, *where* $r, s > -1$ *and let* $\sigma(\phi(b), \phi(a), m) > 0$. *Assume that* $f : A \longrightarrow \mathbb{R}$ *is a twice differentiable function on* A°. *If* $|f''|^q$ *is a generalized beta-preinvex function on* A, $q > 1$, $p^{-1} + q^{-1} = 1$, *then for* $\alpha, k > 0$, *the following inequality for k-fractional Riemann–Liouville integrals holds:*

(3)

$$\begin{aligned}
&\left|A_{\alpha,k}(x;\sigma,\phi,w,m,a,b)\right| \\
&\leq \left(\frac{1}{w+1}\right)^{2+\frac{r+s}{q}} \left(\frac{1}{p\left(\frac{\alpha}{k}+1\right)+1}\right)^{\frac{1}{p}} \frac{k}{(k+\alpha)\sigma(\phi(b),\phi(a),m)} \\
&\quad \times \Bigg\{ |\sigma(\phi(x),\phi(a),m)|^{\frac{\alpha}{k}+2} \Bigg[\frac{m(w+1)^s \cdot {}_2F_1\left(-s, r+1; r+2; \frac{1}{w+1}\right)}{r+1} |f''(\phi(a))|^q \\
&\quad + \frac{(w+1)^r \cdot {}_2F_1\left(-r, s+1; s+2; \frac{1}{w+1}\right)}{s+1} |f''(\phi(x))|^q \Bigg]^{\frac{1}{q}} \\
&\quad + |\sigma(\phi(x),\phi(b),m)|^{\frac{\alpha}{k}+2} \Bigg[\frac{m(w+1)^s \cdot {}_2F_1\left(-s, r+1; r+2; \frac{1}{w+1}\right)}{r+1} |f''(\phi(b))|^q \\
&\quad + \frac{(w+1)^r \cdot {}_2F_1\left(-r, s+1; s+2; \frac{1}{w+1}\right)}{s+1} |f''(\phi(x))|^q \Bigg]^{\frac{1}{q}} \Bigg\}.
\end{aligned}$$

Proof. Suppose that $q > 1$. Using generalized beta-preinvexity of $|f''|^q$, Hölder inequality and taking the modulus, we have

$$\begin{aligned}
&\left|A_{\alpha,k}(x;\sigma,\phi,w,m,a,b)\right| \\
&\leq \frac{|\sigma(\phi(x),\phi(a),m)|^{\frac{\alpha}{k}+2}}{(w+1)^2\left(\frac{\alpha}{k}+1\right)|\sigma(\phi(b),\phi(a),m)|} \\
&\quad \times \int_0^1 t^{\frac{\alpha}{k}+1} \left| f''\left(m\phi(a) + \frac{t}{w+1}\sigma(\phi(x),\phi(a),m)\right)\right| dt \\
&\quad + \frac{|\sigma(\phi(x),\phi(b),m)|^{\frac{\alpha}{k}+2}}{(w+1)^2\left(\frac{\alpha}{k}+1\right)|\sigma(\phi(b),\phi(a),m)|} \\
&\quad \times \int_0^1 (1-t)^{\frac{\alpha}{k}+1} \left| f''\left(m\phi(b) + \frac{t}{w+1}\sigma(\phi(x),\phi(b),m)\right)\right| dt \\
&\leq \frac{|\sigma(\phi(x),\phi(a),m)|^{\frac{\alpha}{k}+2}}{(w+1)^2\left(\frac{\alpha}{k}+1\right)\sigma(\phi(b),\phi(a),m)} \left(\int_0^1 t^{p\left(\frac{\alpha}{k}+1\right)} dt\right)^{\frac{1}{p}}
\end{aligned}$$

$$
\begin{aligned}
&\times\left(\int_0^1\left|f''\left(m\phi(a)+\frac{t}{w+1}\sigma(\phi(x),\phi(a),m)\right)\right|^q dt\right)^{\frac{1}{q}}\\
&+\frac{|\sigma(\phi(x),\phi(b),m)|^{\frac{\alpha}{k}+2}}{(w+1)^2\left(\frac{\alpha}{k}+1\right)\sigma(\phi(b),\phi(a),m)}\left(\int_0^1(1-t)^{p\left(\frac{\alpha}{k}+1\right)}dt\right)^{\frac{1}{p}}\\
&\times\left(\int_0^1\left|f''\left(m\phi(b)+\frac{t}{w+1}\sigma(\phi(x),\phi(b),m)\right)\right|^q dt\right)^{\frac{1}{q}}\\
\leq&\frac{|\sigma(\phi(x),\phi(a),m)|^{\frac{\alpha}{k}+2}}{(w+1)^2\left(\frac{\alpha}{k}+1\right)\sigma(\phi(b),\phi(a),m)}\left(\int_0^1 t^{p\left(\frac{\alpha}{k}+1\right)}dt\right)^{\frac{1}{p}}\\
&\times\left[\int_0^1\left(m\left(\frac{t}{w+1}\right)^r\left(1-\frac{t}{w+1}\right)^s|f''(\phi(a))|^q\right.\right.\\
&\left.\left.+\left(\frac{t}{w+1}\right)^s\left(1-\frac{t}{w+1}\right)^r|f''(\phi(x))|^q\right)dt\right]^{\frac{1}{q}}\\
&+\frac{|\sigma(\phi(x),\phi(b),m)|^{\frac{\alpha}{k}+2}}{(w+1)^2\left(\frac{\alpha}{k}+1\right)\sigma(\phi(b),\phi(a),m)}\left(\int_0^1(1-t)^{p\left(\frac{\alpha}{k}+1\right)}dt\right)^{\frac{1}{p}}\\
&\times\left[\int_0^1\left(m\left(\frac{t}{w+1}\right)^r\left(1-\frac{t}{w+1}\right)^s|f''(\phi(b))|^q\right.\right.\\
&\left.\left.+\left(\frac{t}{w+1}\right)^s\left(1-\frac{t}{w+1}\right)^r|f''(\phi(x))|^q\right)dt\right]^{\frac{1}{q}}\\
=&\left(\frac{1}{w+1}\right)^{2+\frac{r+s}{q}}\left(\frac{1}{p\left(\frac{\alpha}{k}+1\right)+1}\right)^{\frac{1}{p}}\frac{k}{(k+\alpha)\sigma(\phi(b),\phi(a),m)}\\
&\times\left\{|\sigma(\phi(x),\phi(a),m)|^{\frac{\alpha}{k}+2}\left[\frac{m(w+1)^s\cdot{}_2F_1\left(-s,r+1;r+2;\frac{1}{w+1}\right)}{r+1}|f''(\phi(a))|^q\right.\right.\\
&\left.+\frac{(w+1)^r\cdot{}_2F_1\left(-r,s+1;s+2;\frac{1}{w+1}\right)}{s+1}|f''(\phi(x))|^q\right]^{\frac{1}{q}}\\
&+|\sigma(\phi(x),\phi(b),m)|^{\frac{\alpha}{k}+2}\left[\frac{m(w+1)^s\cdot{}_2F_1\left(-s,r+1;r+2;\frac{1}{w+1}\right)}{r+1}|f''(\phi(b))|^q\right.\\
&\left.\left.+\frac{(w+1)^r\cdot{}_2F_1\left(-r,s+1;s+2;\frac{1}{w+1}\right)}{s+1}|f''(\phi(x))|^q\right]^{\frac{1}{q}}\right\}.
\end{aligned}
$$

The proof of Theorem 4 is completed. □

Corollary 3. *Under the conditions of Theorem 4, if we choose $w = 0$, $m = k = 1$ and $\sigma(\phi(y), \phi(x), m) = \phi(y) - m\phi(x)$, $\forall x, y \in I$, then we get the following generalized Hermite–Hadamard type inequality for fractional integrals:*

$$\Bigg|\frac{(\phi(x)-\phi(a))^{\alpha+1} f'(\phi(x)) - (\phi(b)-\phi(x))^{\alpha+1} f'(\phi(b))}{(\alpha+1)(\phi(b)-\phi(a))}$$

$$- \frac{(\phi(x)-\phi(a))^{\alpha} f(\phi(x)) + (\phi(b)-\phi(x))^{\alpha} f(\phi(b))}{\phi(b)-\phi(a)}$$

$$+ \frac{\Gamma(\alpha+1)}{(\phi(b)-\phi(a))} \times \left[J^{\alpha}_{\phi(x)^-} f(\phi(a)) + J^{\alpha}_{\phi(b)^+} f(\phi(x))\right]\Bigg|$$

$$\leq \frac{1}{(\alpha+1)(\phi(b)-\phi(a))} \left(\frac{1}{p(\alpha+1)+1}\right)^{\frac{1}{p}} \Bigg\{(\phi(x)-\phi(a))^{\alpha+2}$$

$$\times \left[\frac{{}_2F_1(-s, r+1; r+2; 1)}{r+1} |f''(\phi(a))|^q + \frac{{}_2F_1(-r, s+1; s+2; 1)}{s+1} |f''(\phi(x))|^q\right]^{\frac{1}{q}}$$

$$+ (\phi(b)-\phi(x))^{\alpha+2}$$

$$\times \left[\frac{{}_2F_1(-s, r+1; r+2; 1)}{r+1} |f''(\phi(b))|^q + \frac{{}_2F_1(-r, s+1; s+2; 1)}{s+1} |f''(\phi(x))|^q\right]^{\frac{1}{q}}\Bigg\}.$$

Theorem 5. *Let $\phi : I \longrightarrow A$ be a continuous function. Suppose that $A = [m\phi(a), m\phi(a) + \sigma(\phi(b), \phi(a), m)] \subseteq \mathbb{R}$ is an open m-invex subset with respect to $\sigma : A \times A \times (0,1] \longrightarrow \mathbb{R}$ for some fixed $m \in (0,1]$ and $\forall w \in [0,1]$, where $r, s > -1$ and let $\sigma(\phi(b), \phi(a), m) > 0$. Assume that $f : A \longrightarrow \mathbb{R}$ is a twice differentiable function on A°. If $|f''|^q$ is a generalized beta-preinvex function on A, $q \geq 1$, then for $\alpha, k > 0$, the following inequality for k-fractional Riemann–Liouville integrals holds:*

$$\begin{aligned}
&\left|A_{\alpha,k}(x; \sigma, \phi, w, m, a, b)\right| \\
&\leq \left(\frac{1}{w+1}\right)^{2+\frac{r+s}{q}} \frac{k^{2-\frac{1}{q}}}{(k+\alpha)(2k+\alpha)^{1-\frac{1}{q}} \sigma(\phi(b), \phi(a), m)} \\
&\quad \times \Bigg\{|\sigma(\phi(x), \phi(a), m)|^{\frac{\alpha}{k}+2} \\
&\quad \times \Bigg[\frac{m(w+1)^s \cdot {}_2F_1\left(-s, \frac{\alpha}{k}+r+2; \frac{\alpha}{k}+r+3; \frac{1}{w+1}\right)}{\frac{\alpha}{k}+r+2} |f''(\phi(a))|^q \\
&\quad + \frac{(w+1)^r \cdot {}_2F_1\left(-r, \frac{\alpha}{k}+s+2; \frac{\alpha}{k}+s+3; \frac{1}{w+1}\right)}{\frac{\alpha}{k}+s+2} |f''(\phi(x))|^q\Bigg]^{\frac{1}{q}}
\end{aligned} \tag{4}$$

$$+ |\sigma(\phi(x), \phi(b), m)|^{\frac{\alpha}{k}+2}$$
$$\times \Bigg[m(w+1)^s \beta\left(r+1, \frac{\alpha}{k}+2\right) \cdot {}_2F_1\left(-s, r+1; \frac{\alpha}{k}+r+3; \frac{1}{w+1}\right) |f''(\phi(b))|^q$$
$$+ (w+1)^r \beta\left(s+1, \frac{\alpha}{k}+2\right) \cdot {}_2F_1\left(-r, s+1; \frac{\alpha}{k}+s+3; \frac{1}{w+1}\right) |f''(\phi(x))|^q \Bigg]^{\frac{1}{q}} \Bigg\}.$$

Proof. Suppose that $q \geq 1$. Using the generalized beta-preinvexity of $|f''|^q$, the well-known power mean inequality and taking the modulus, we have

$$\begin{aligned}
&\left|A_{\alpha,k}(x; \sigma, \phi, w, m, a, b)\right| \\
&\leq \frac{|\sigma(\phi(x), \phi(a), m)|^{\frac{\alpha}{k}+2}}{(w+1)^2 \left(\frac{\alpha}{k}+1\right) |\sigma(\phi(b), \phi(a), m)|} \\
&\quad \times \int_0^1 t^{\frac{\alpha}{k}+1} \left| f''\left(m\phi(a) + \frac{t}{w+1}\sigma(\phi(x), \phi(a), m)\right)\right| dt \\
&\quad + \frac{|\sigma(\phi(x), \phi(b), m)|^{\frac{\alpha}{k}+2}}{(w+1)^2 \left(\frac{\alpha}{k}+1\right) |\sigma(\phi(b), \phi(a), m)|} \\
&\quad \times \int_0^1 (1-t)^{\frac{\alpha}{k}+1} \left| f''\left(m\phi(b) + \frac{t}{w+1}\sigma(\phi(x), \phi(b), m)\right)\right| dt \\
&\leq \frac{|\sigma(\phi(x), \phi(a), m)|^{\frac{\alpha}{k}+2}}{(w+1)^2 \left(\frac{\alpha}{k}+1\right) \sigma(\phi(b), \phi(a), m)} \left(\int_0^1 t^{\frac{\alpha}{k}+1} dt\right)^{1-\frac{1}{q}} \\
&\quad \times \left(\int_0^1 t^{\frac{\alpha}{k}+1} \left| f''\left(m\phi(a) + \frac{t}{w+1}\sigma(\phi(x), \phi(a), m)\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\quad + \frac{|\sigma(\phi(x), \phi(b), m)|^{\frac{\alpha}{k}+2}}{(w+1)^2 \left(\frac{\alpha}{k}+1\right) \sigma(\phi(b), \phi(a), m)} \left(\int_0^1 (1-t)^{\frac{\alpha}{k}+1} dt\right)^{1-\frac{1}{q}} \\
&\quad \times \left(\int_0^1 (1-t)^{\frac{\alpha}{k}+1} \left| f''\left(m\phi(b) + \frac{t}{w+1}\sigma(\phi(x), \phi(b), m)\right)\right|^q dt\right)^{\frac{1}{q}} \\
&\leq \frac{|\sigma(\phi(x), \phi(a), m)|^{\frac{\alpha}{k}+2}}{(w+1)^2 \left(\frac{\alpha}{k}+1\right) \sigma(\phi(b), \phi(a), m)} \left(\int_0^1 t^{\frac{\alpha}{k}+1} dt\right)^{1-\frac{1}{q}} \\
&\quad \times \Bigg[\int_0^1 t^{\frac{\alpha}{k}+1} \Bigg(m\left(\frac{t}{w+1}\right)^r \left(1-\frac{t}{w+1}\right)^s |f''(\phi(a))|^q \\
&\quad + \left(\frac{t}{w+1}\right)^s \left(1-\frac{t}{w+1}\right)^r |f''(\phi(x))|^q \Bigg) dt\Bigg]^{\frac{1}{q}}
\end{aligned}$$

$$+\frac{|\sigma(\phi(x),\phi(b),m)|^{\frac{\alpha}{k}+2}}{(w+1)^2\left(\frac{\alpha}{k}+1\right)\sigma(\phi(b),\phi(a),m)}\left(\int_0^1(1-t)^{\frac{\alpha}{k}+1}dt\right)^{1-\frac{1}{q}}$$

$$\times\left[\int_0^1(1-t)^{\frac{\alpha}{k}+1}\left(m\left(\frac{t}{w+1}\right)^r\left(1-\frac{t}{w+1}\right)^s|f''(\phi(b))|^q\right.\right.$$

$$\left.\left.+\left(\frac{t}{w+1}\right)^s\left(1-\frac{t}{w+1}\right)^r|f''(\phi(x))|^q\right)dt\right]^{\frac{1}{q}}$$

$$=\left(\frac{1}{w+1}\right)^{2+\frac{r+s}{q}}\frac{k^{2-\frac{1}{q}}}{(k+\alpha)(2k+\alpha)^{1-\frac{1}{q}}\sigma(\phi(b),\phi(a),m)}$$

$$\times\left\{|\sigma(\phi(x),\phi(a),m)|^{\frac{\alpha}{k}+2}\right.$$

$$\times\left[\frac{m(w+1)^s\cdot{}_2F_1\left(-s,\frac{\alpha}{k}+r+2;\frac{\alpha}{k}+r+3;\frac{1}{w+1}\right)}{\frac{\alpha}{k}+r+2}|f''(\phi(a))|^q\right.$$

$$\left.+\frac{(w+1)^r\cdot{}_2F_1\left(-r,\frac{\alpha}{k}+s+2;\frac{\alpha}{k}+s+3;\frac{1}{w+1}\right)}{\frac{\alpha}{k}+s+2}|f''(\phi(x))|^q\right]^{\frac{1}{q}}$$

$$+|\sigma(\phi(x),\phi(b),m)|^{\frac{\alpha}{k}+2}$$

$$\times\left[m(w+1)^s\beta\left(r+1,\frac{\alpha}{k}+2\right)\cdot{}_2F_1\left(-s,r+1;\frac{\alpha}{k}+r+3;\frac{1}{w+1}\right)|f''(\phi(b))|^q\right.$$

$$\left.\left.+(w+1)^r\beta\left(s+1,\frac{\alpha}{k}+2\right)\cdot{}_2F_1\left(-r,s+1;\frac{\alpha}{k}+s+3;\frac{1}{w+1}\right)|f''(\phi(x))|^q\right]^{\frac{1}{q}}\right\}.$$

The proof of Theorem 5 is completed. □

Corollary 4. *Under the conditions of Theorem 5, if we choose $w=0$, $m=k=1$ and $\sigma(\phi(y),\phi(x),m)=\phi(y)-m\phi(x)$, $\forall x,y\in I$, then we get the following generalized Hermite–Hadamard type inequality for fractional integrals:*

$$\left|\frac{(\phi(x)-\phi(a))^{\alpha+1}f'(\phi(x))-(\phi(b)-\phi(x))^{\alpha+1}f'(\phi(b))}{(\alpha+1)(\phi(b)-\phi(a))}\right.$$

$$-\frac{(\phi(x)-\phi(a))^{\alpha}f(\phi(x))+(\phi(b)-\phi(x))^{\alpha}f(\phi(b))}{\phi(b)-\phi(a)}$$

$$\left.+\frac{\Gamma(\alpha+1)}{(\phi(b)-\phi(a))}\times\left[J^{\alpha}_{\phi(x)-}f(\phi(a))+J^{\alpha}_{\phi(b)+}f(\phi(x))\right]\right|$$

$$\leq \frac{1}{(\alpha+1)(\alpha+2)^{1-\frac{1}{q}}(\phi(b)-\phi(a))}$$
$$\times\Bigg\{(\phi(x)-\phi(a))^{\alpha+2}\Bigg[\frac{{}_2F_1(-s,\alpha+r+2;\alpha+r+3;1)}{\alpha+r+2}|f''(\phi(a))|^q$$
$$+\frac{{}_2F_1(-r,\alpha+s+2;\alpha+s+3;1)}{\alpha+s+2}|f''(\phi(x))|^q\Bigg]^{\frac{1}{q}}$$
$$+(\phi(b)-\phi(x))^{\alpha+2}\Big[\beta(r+1,\alpha+2)\cdot{}_2F_1(-s,r+1;\alpha+r+3;1)|f''(\phi(b))|^q$$
$$+\beta(s+1,\alpha+2)\cdot{}_2F_1(-r,s+1;\alpha+s+3;1)|f''(\phi(x))|^q\Big]^{\frac{1}{q}}\Bigg\}.$$

Remark 3. For $M>0$, if $|f''|\leq M$, then by Theorem 4 and 5, we can get some special kinds of Hermite–Hadamard-type inequalities via k-fractional Riemann–Liouville integrals. For $k=1$, we obtain special kinds of Hermite–Hadamard-type inequalities via Riemann–Liouville integrals. Also, for different choices of w, for example $w=\frac{1}{2},\frac{1}{3},1$, by Theorem 4 and 5 we can get some interesting integral inequalities of these types.

4. Applications to special means

Definition 12. ([18]) A function $M:\mathbb{R}_+^2\longrightarrow\mathbb{R}_+$, is called a Mean function if it has the following properties:

(1) Homogeneity: $M(ax,ay)=aM(x,y)$, for all $a>0$,
(2) Symmetry: $M(x,y)=M(y,x)$,
(3) Reflexivity: $M(x,x)=x$,
(4) Monotonicity: If $x\leq x'$ and $y\leq y'$, then $M(x,y)\leq M(x',y')$,
(5) Internality: $\min\{x,y\}\leq M(x,y)\leq\max\{x,y\}$.

We consider some means for arbitrary positive real numbers α,β $(\alpha\neq\beta)$.

(1) The arithmetic mean:
$$A:=A(\alpha,\beta)=\frac{\alpha+\beta}{2}.$$
(2) The geometric mean:
$$G:=G(\alpha,\beta)=\sqrt{\alpha\beta}.$$
(3) The harmonic mean:
$$H:=H(\alpha,\beta)=\frac{2}{\frac{1}{\alpha}+\frac{1}{\beta}}.$$

(4) The power mean:

$$P_r := P_r(\alpha, \beta) = \left(\frac{\alpha^r + \beta^r}{2} \right)^{\frac{1}{r}}, \quad r \geq 1.$$

(5) The identic mean:

$$I := I(\alpha, \beta) = \begin{cases} \frac{1}{e} \left(\frac{\beta^\beta}{\alpha^\alpha} \right), & \alpha \neq \beta; \\ \alpha, & \alpha = \beta. \end{cases}$$

(6) The logarithmic mean:

$$L := L(\alpha, \beta) = \frac{\beta - \alpha}{\ln \beta - \ln \alpha}.$$

(7) The generalized log-mean:

$$L_p := L_p(\alpha, \beta) = \left[\frac{\beta^{p+1} - \alpha^{p+1}}{(p+1)(\beta - \alpha)} \right]^{\frac{1}{p}}; \quad p \in \mathbb{R} \setminus \{-1, 0\}.$$

(8) The weighted p-power mean:

$$M_p \begin{pmatrix} \alpha_1, \alpha_2, \ldots, \alpha_n \\ u_1, u_2, \ldots, u_n \end{pmatrix} = \left(\sum_{i=1}^{n} \alpha_i u_i^p \right)^{\frac{1}{p}},$$

where $0 \leq \alpha_i \leq 1$, $u_i > 0$ $(i = 1, 2, \ldots, n)$ with $\sum_{i=1}^{n} \alpha_i = 1$.

It is well known that L_p is nondecreasing over $p \in \mathbb{R}$ with $L_{-1} := L$ and $L_0 := I$. In particular, we have the following inequality $H \leq G \leq L \leq I \leq A$. Now, let a and b be positive real numbers such that $a < b$. Consider the function $\mathcal{M} := \mathcal{M}(\phi(a), \phi(b)) : [\phi(a), \phi(a) + \sigma(\phi(b), \phi(a))] \times [\phi(a), \phi(a) + \sigma(\phi(b), \phi(a))] \longrightarrow \mathbb{R}_+$, which is one of the above mentioned means and let $\phi : I \longrightarrow A$ be a continuous function, therefore one can obtain various inequalities using the results of Section 3 for these means as follows. Replacing $\sigma(\phi(y), \phi(x), m)$ with $\sigma(\phi(y), \phi(x))$ and setting $\sigma(\phi(a), \phi(x)) = \mathcal{M}(\phi(a), \phi(x))$, $\sigma(\phi(b), \phi(x)) = \mathcal{M}(\phi(b), \phi(x))$, $\forall x \in I$, for value $m = 1$ in (3) and (4), one can obtain the following interesting inequalities involving means:

(5)

$$\begin{aligned}
&\left|A_{\alpha,k}(x;\mathcal{M}(\cdot,\cdot),\phi,w,1,a,b)\right| \\
&=\left|\frac{\mathcal{M}^{\frac{\alpha}{k}+1}(\phi(a),\phi(x))f'\left(\phi(a)+\frac{\mathcal{M}(\phi(a),\phi(x))}{r+1}\right)-\mathcal{M}^{\frac{\alpha}{k}+1}(\phi(b),\phi(x))f'(\phi(b))}{(r+1)\left(\frac{\alpha}{k}+1\right)\mathcal{M}(\phi(a),\phi(b))}\right. \\
&\quad-\frac{\mathcal{M}^{\frac{\alpha}{k}}(\phi(a),\phi(x))f\left(\phi(a)+\frac{\mathcal{M}(\phi(a),\phi(x))}{r+1}\right)+\mathcal{M}^{\frac{\alpha}{k}}(\phi(b),\phi(x))f(\phi(b))}{\mathcal{M}(\phi(a),\phi(b))} \\
&\quad+\frac{(r+1)^{\frac{\alpha}{k}}\Gamma_k(\alpha+k)}{\mathcal{M}(\phi(a),\phi(b))} \\
&\quad\left.\times\left[I^{\alpha,k}_{\phi(b)^+}f\left(\phi(b)+\frac{\mathcal{M}(\phi(b),\phi(x))}{r+1}\right)+I^{\alpha,k}_{\left(\phi(a)+\frac{\mathcal{M}(\phi(a),\phi(x))}{r+1}\right)^-}f(\phi(a))\right]\right| \\
&\le\left(\frac{1}{w+1}\right)^{2+\frac{r+s}{q}}\left(\frac{1}{p\left(\frac{\alpha}{k}+1\right)+1}\right)^{\frac{1}{p}}\frac{k}{(k+\alpha)\mathcal{M}(\phi(a),\phi(b))} \\
&\quad\times\left\{\mathcal{M}(\phi(a),\phi(x))^{\frac{\alpha}{k}+2}\left[\frac{(w+1)^s\cdot{}_2F_1\left(-s,r+1;r+2;\frac{1}{w+1}\right)}{r+1}|f''(\phi(a))|^q\right.\right. \\
&\quad\left.+\frac{(w+1)^r\cdot{}_2F_1\left(-r,s+1;s+2;\frac{1}{w+1}\right)}{s+1}|f''(\phi(x))|^q\right]^{\frac{1}{q}} \\
&\quad+\mathcal{M}(\phi(b),\phi(x))^{\frac{\alpha}{k}+2}\left[\frac{(w+1)^s\cdot{}_2F_1\left(-s,r+1;r+2;\frac{1}{w+1}\right)}{r+1}|f''(\phi(b))|^q\right. \\
&\quad\left.\left.+\frac{(w+1)^r\cdot{}_2F_1\left(-r,s+1;s+2;\frac{1}{w+1}\right)}{s+1}|f''(\phi(x))|^q\right]^{\frac{1}{q}}\right\},
\end{aligned}$$

and

(6)

$$\begin{aligned}
&\left|A_{\alpha,k}(x;\mathcal{M}(\cdot,\cdot),\phi,w,1,a,b)\right| \\
&\le\left(\frac{1}{w+1}\right)^{2+\frac{r+s}{q}}\frac{k^{2-\frac{1}{q}}}{(k+\alpha)(2k+\alpha)^{1-\frac{1}{q}}\mathcal{M}(\phi(a),\phi(b))} \\
&\quad\times\left\{\mathcal{M}^{\frac{\alpha}{k}+2}(\phi(a),\phi(x))\left[\frac{(w+1)^s\cdot{}_2F_1\left(-s,\frac{\alpha}{k}+r+2;\frac{\alpha}{k}+r+3;\frac{1}{w+1}\right)}{\frac{\alpha}{k}+r+2}|f''(\phi(a))|^q\right.\right.
\end{aligned}$$

$$+\frac{(w+1)^r\cdot{}_2F_1\left(-r,\frac{\alpha}{k}+s+2;\frac{\alpha}{k}+s+3;\frac{1}{w+1}\right)}{\frac{\alpha}{k}+s+2}|f''(\phi(x))|^q\Bigg]^{\frac{1}{q}}$$

$$+\mathcal{M}^{\frac{\alpha}{k}+2}(\phi(b),\phi(x))$$

$$\times\Bigg[(w+1)^s\beta\Big(r+1,\frac{\alpha}{k}+2\Big)\cdot{}_2F_1\Big(-s,r+1;\frac{\alpha}{k}+r+3;\frac{1}{w+1}\Big)|f''(\phi(b))|^q$$

$$+(w+1)^r\beta\Big(s+1,\frac{\alpha}{k}+2\Big)\cdot{}_2F_1\Big(-r,s+1;\frac{\alpha}{k}+s+3;\frac{1}{w+1}\Big)|f''(\phi(x))|^q\Bigg]^{\frac{1}{q}}\Bigg\}.$$

Letting $\mathcal{M}(\phi(x),\phi(y)) := A, G, H, P_r, I, L, L_p, M_p,\ \forall x, y \in I$, in (5) and (6), we get the inequalities involving means for a particular choices of a twice differentiable generalized beta-preinvex function f. The details are left to the interested reader.

5. Conclusions

We have considered and investigated the class of generalized beta-preinvex functions. Some new integral inequalities for the left hand side of the Gauss-Jacobi type quadrature formula involving generalized beta-preinvex functions are proved. Moreover, using new integral identity, some Hermite–Hadamard type inequalities for generalized beta-preinvex functions that are twice differentiable via k-fractional integrals are established. At the end, some applications to special means are given. These general inequalities give us some new estimates for Hermite–Hadamard type k-fractional integral inequalities. We conclude that our methods considered here may be a stimulant for further investigations concerning Hermite–Hadamard type integral inequalities for various kinds of convex and preinvex functions. We believe that our results can be treated using quantum calculus as well.

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Received April 05, 2017.

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