ON SOME RESULTS OBTAINED BY THE QUATERNION ANALYSIS RESPECTING THE INSCRIPTION OF “GAUCHE” POLYGONS IN SURFACES OF THE SECOND ORDER

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By Sir William R. Hamilton.

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Sir William Rowan Hamilton communicated to the Academy some results, obtained by the quaternion analysis, respecting the *inscription of gauche polygons in surfaces of the second order.*

If it be required to inscribe a rectilinear polygon $\text{P}, \text{P}_1, \text{P}_2, \ldots \text{P}_{n-1}$ in such a surface, under the conditions that its n successive sides $\text{PP}_1, \text{P}_1\text{P}_2, \ldots \text{P}_{n-1}\text{P}$, shall pass respectively through n given points, $\text{A}_1, \text{A}_2, \ldots \text{A}_n$, the analysis of Sir W. R. H. conducts to *one*, or to *two real lines*, as containing the first corner P, according as the number n of sides is *odd* or *even*: while, in the latter of these two cases, the two real right lines thus found are *reciprocal polars* of each other, with reference to the surface in which the polygon is to be inscribed. Thus, for the inscription of a plane triangle, or of a gauche pentagon, heptagon, &c., in a surface of the second order, where three, five, seven, &c, points are given upon its sides, a single right line is found, which may or may not intersect the surface; and the problem of inscription admits generally of two real *or* of two imaginary solutions. But for the inscription of a gauche quadrilateral, hexagon, octagon, &c., when four, six, eight, &c. points are given on its successive sides, two real right lines are found, which (as above stated) are polars of each other; and therefore, if the surface be an ellipsoid, or a hyperboloid of *two* sheets, the problem admits generally of two real *and* of two imaginary solutions: while if the surface be a hyperboloid of *one* sheet, the four solutions are then, in general, together real, or together imaginary.

When a gauche polygon, or polygon with $2m + 1$ sides, is to be inscribed in an ellipsoid or in a double-sheeted hyperboloid, and when the single straight line, found as above, lies wholly outside the surface, so as to give two imaginary solutions of the problem as at first proposed, this line is still not useless geometrically; for its reciprocal polar intersects the surface in two real points, of which each is the first corner of an inscribed decagon, or polygon with $4m + 2$ sides, whose $2m + 1$ pairs of *opposite sides* intersect each other respectively in $2m + 1$ given points, $\text{A}_1, \text{A}_2, \ldots \text{A}_{2m+1}$. Thus when, in the well-known problem of inscribing a triangle in a plane conic, whose sides shall pass through three given points, the known rectilinear locus of the first corner is found to have no real intersection with the conic, so that the problem, as usually viewed, admits of no real solution, and that the inscription of the *triangle* becomes geometrically *impossible*; we have only to conceive an ellipsoid, or a double-sheeted hyperboloid, to be so constructed as to contain the given conic upon its surface; and then to take, with respect to this surface, the polar of this known right line, in order to obtain two *real* or geometrically possible solutions of *another* problem, not less interesting: since this rectilinear polar will cut the surface in two real points, of which each is the first corner of an *inscribed gauche hexagon* whose *opposite sides intersect* each other in the three points proposed. (It may be noticed that the three *diagonals* of this gauche hexagon, or the three right lines joining each corner to the opposite one, intersect each other in *one common point*, namely, in the pole of the given plane.)

If we seek to inscribe a polygon of $4m$ sides in a surface of the second order, under the condition that its opposite sides shall intersect respectively in $2m$ given points, the quaternion analysis conducts generally to two polar right lines, as loci of the first corner, which lines are the same with those that would be otherwise found as loci of the first corner of an inscribed polygon of $2m$ sides, passing respectively through the $2m$ given points. Thus, *in general*, the polygon of $4m$ sides, found as above, is merely the polygon of $2m$ sides, with *each side twice traversed* by the motion of a point along its perimeter. But if a certain *condition* be satisfied, by a certain *arrangement of the* $2m$ *given points* in space; namely, if the last point A_{2m} be on that real right line which is the locus of the first corner of a real or imaginary inscribed polygon of $2m-1$ sides, which pass respectively through the first $2m-1$ given points $\mathrm{A}_1, \ldots \mathrm{A}_{2m-1}$; then the inscribed polygon of $4m$ distinct sides becomes not only possible but *indeterminate*, its first corner being in this case allowed to take *any position on the surface*. For example, if the two triangles $\mathrm{P}'\mathrm{P}_1'\mathrm{P}_2{}'$, $\mathrm{P}''\mathrm{P}_1''\mathrm{P}_2{}''$ be inscribed in a conic, so that the corresponding sides $\mathrm{P}'\mathrm{P}_1'$ and $\mathrm{P}''\mathrm{P}_1''$ intersect each other in A_1; $\mathrm{P}_1'\mathrm{P}_2'$ and $\mathrm{P}_1''\mathrm{P}_2''$ in A_2; and $\mathrm{P}_2'\mathrm{P}'$ and $\mathrm{P}_2''\mathrm{P}''$ in A_3; and if we take a fourth point A_4 on the right line $\mathrm{P}'\mathrm{P}''$, and conceive any surface of the second order constructed so as to contain the given conic; then *any point* P, on this surface, is fit to be the first corner of a plane or gauche *octagon*, $\mathrm{P}\,\mathrm{P}_1 \ldots \mathrm{P}_7$, inscribed in the surface, so that the first and fifth sides PP_1, $\mathrm{P}_4\mathrm{P}_5$ shall intersect in A_1; the second and sixth sides in A_2; the third and seventh sides in A_3; and the fourth and eighth in A_4. And generally if $2m$ given points be points of intersection of opposite sides of *any one* inscribed polygon of $4m$ sides, the *same* $2m$ *points* are then fit to be intersections of opposite sides of *infinitely many other* inscribed polygons, plane or gauche, of $4m$ sides. A very elementary example is furnished by an inscribed plane quadrilateral, of which the two points of meeting of opposite sides are well known to be *conjugate*, relatively to the conic or to the surface, and are adapted to be the points of meeting of opposite sides of infinitely many other inscribed quadrilaterals.

When *all the sides but one*, of an inscribed gauche polygon, pass through given points, the *remaining side* may be said *generally* to be *doubly tangent* to a real or imaginary *surface of the fourth order*, which separates itself into *two* real or imaginary *surfaces of the second order*, having real or imaginary *double contact* with the original surface of the second order, and with each other. If the original surface be an ellipsoid (E), and if the number of sides of the inscribed polygon, $\mathrm{PP}_1, \ldots \mathrm{P}_{2m}$, be odd, $= 2m+1$, so that the number of fixed points $\mathrm{A}_1, \ldots \mathrm{A}_{2m}$ is even, $= 2m$, then the two surfaces enveloped by the last side $\mathrm{P}_{2m}\mathrm{P}$ are a *real inscribed ellipsoid* (E'), and a *real exscribed hyperboloid of two sheets* (E''); and these three surfaces (E) (E') (E'') touch each other at the *two real points* B, B', which are the first corners of two inscribed polygons $\mathrm{BB}_1, \ldots \mathrm{B}_{2m-1}$ and $\mathrm{B}'\mathrm{B}_1', \ldots \mathrm{B}_{2m-1}'$, whose $2m$ sides pass respectively through the $2m$ given points (A). If these three surfaces of the second order be cut by any three planes parallel to either of the two common tangent planes at B and B', the sections are three *similar and similarly placed ellipses*; thus B and B' are two of the four *umbilics* of the ellipsoid (E'), and also of the hyperboloid (E''), when the original surface E is a *sphere*. The *closing chords* $P_{2m}P$ touch a series of real *curves* (C') on (E'), and *also* another series of real curves (C'') on (E''), which curves are the *arêtes de rebroussement* of two series of *developable surfaces*, (D') and (D''), into which latter surfaces the closing chords arrange themselves; but these two sets of developable surfaces are *not* generally *rectangular* to each other, and consequently the closing chords themselves are *not generally perpendicular to any one common surface*. However, when (E) is a sphere, the developable surfaces cut it in two series of curves, (F'), (F''), which everywhere cross each other at right angles; and generally at any point P on (E), the tangents to the two curves (F') and (F'') are parallel to two conjugate semidiameters.

The *centres* of the three surfaces of the second order are placed on *one straight line*; and every closing chord $P_{2m}P$ is *cut harmonically* at the points where it touches the two surfaces (E'), (E''), or the two curves (C'), (C''), which are the *arêtes* of the two developable surfaces (D'), (D''), passing through that chord $P_{2m}P$. In a certain class of *cases* the three surfaces (E), (E'), (E'') are all of *revolution*, round one common axis; and when this happens, the curves (C'), (C''), (F'), (F'') are certain *spires* upon these surfaces, having this *common character*, that for any one such spire *equal rotations* round the axis give *equal anharmonic ratios*: or that, more fully, if on a spire (C'), for example, there be taken two pairs of points C'_1, C'_2 and C'_3, C'_4, and if these be projected on the axis BB' in points G'_1, G'_2 and G'_3, G'_4, then the rectangle $BG'_1 \,.\, G'_2B'$ will be to the rectangle $BG'_2 \,.\, G'_1B'$, as $BG'_3 \,.\, G'_4B'$, to $BG'_4 \,.\, G'_3B'$, if the dihedral angle $C'_1BB'C'_2$ be equal to the dihedral angle $C'_3BB'C'_4$. In another extensive class of cases the hyperboloid or two sheets (E'') reduces itself to a pair of planes, touching the given ellipsoid (E) in the points B and B'; and then the prolongations of the closing chords, $P_{2m}P$, all meet the right line of intersection of these two tangent planes: or the inscribed ellipsoid (E') may reduce itself to the right line BB', which is, in that case, crossed by all the closing chords. For example, if the first four sides of an inscribed gauche pentagon pass respectively through four given points, which are all in one common plane, then the fifth side of the pentagon intersects a fixed right line in that plane.

An example of *imaginary envelopes* is suggested by the problem of inscribing a gauche quadrilateral, hexagon, or polygon of $2m$ sides in an ellipsoid, all the sides but the last being obliged to pass through fixed points. In this problem the *last side* may be said to touch two imaginary surfaces of the second order, which intersect each other in two real or imaginary conics, situated in two real planes; and when these two conics are real, they touch the original ellipsoid in two real and common points, which are the two positions of the first corner of an inscribed polygon, whose sides pass through the $2m - 1$ fixed points. Every rectilinear tangent to *either* conic is a closing chord $P_{2m-1}P$; but no position of that closing chord, which is not thus a tangent to one or other of these conics, is intersected *anywhere* by *any* infinitely near chord of the system. These results were illustrated by an example, in which there were three given points; one conic was the known envelope of the fourth side of a plane inscribed quadrilateral; and this was found to be the *ellipse de gorge* of a certain single-sheeted hyperboloid, a certain section of which hyperboloid, by a plane perpendicular to the plane of the ellipse, gave the *hyperbola* which was, in this example, the *other* real conic, and was thus situated in a plane *perpendicular* to the plane of the ellipse. And to illustrate the *imaginary* character of the *enveloped surfaces*, or the general non-intersection (in this example) of infinitely near positions of the closing chords in space, *one* such chord was selected, and it was shown that all the infinitely near chords, which made with *this* chord equal and infinitesimal angles, were generatrices (of one common system) of an infinitely thin and single-sheeted hyperboloid.

Conceive that any rectilinear polygon of n sides, $\mathrm{BB}_1, \ldots \mathrm{B}_{n-1}$, has been inscribed in any surface of the second order, and that n points $\mathrm{A}_1, \ldots \mathrm{A}_n$ have been assumed on its n sides, $\mathrm{BB}_1, \ldots \mathrm{B}_{n-1}\mathrm{B}$. Take then at pleasure any point P upon the same surface, and draw the chords $\mathrm{PA}_1\mathrm{P}_1, \ldots \mathrm{P}_{n-1}\mathrm{A}_n\mathrm{P}_n$, passing respectively through the n points (A). Again begin with P_n, and draw, through the same n points (A), n other successive chords, $\mathrm{P}_n\mathrm{A}_1\mathrm{P}_{n+1}, \ldots \mathrm{P}_{2n-1}\mathrm{A}_n\mathrm{P}_{2n}$. Again, draw the n chords, $\mathrm{P}_{2n}\mathrm{A}_1\mathrm{P}_{2n+1}, \ldots \mathrm{P}_{3n-1}\mathrm{A}_n\mathrm{P}_{3n}$. Draw tangent planes at P_n and P_{2n}, meeting the two new chords PP_{2n} and $\mathrm{P}_n\mathrm{P}_{3n}$ in points R, R′; and draw any rectilinear tangent BC at B. Then one or other of the two following theorems will hold good, according as n is an *odd* or *even* number. When n is *odd*, the three points BRR′ will be situated in one straight line. When n is *even*, the three pyramids which have BC for a common edge, and have for their edges respectively opposite thereto the three chords PP_{2n}, $\mathrm{P}_{2n}\mathrm{P}_n$, $\mathrm{P}_n\mathrm{P}_{3n}$, being divided respectively by the squares of those three chords, and multiplied by the squares of the three respectively parallel semidiameters of the surface, and being also taken with algebraic signs which it is easy to determine, have their sum equal to zero. Both theorems conduct to a form of Poncelet's construction (the present writer's knowledge of which is derived chiefly from the valuable work on Conic Sections, by the Rev. George Salmon, F. T. C. D.), when applied to the problem of inscribing a polygon in a plane conic: and the second theorem may easily be stated generally under a *graphic* intesd of a *metric* form.

The analysis by which these results, and others connected with them, have been obtained, appears to the author to be sufficiently simple, as least if regard be had to the novelty and difficulty of some of the questions to which it has been thus applied; but he conceives that it would occupy too large a space in the Proceedings, if he were to give any account of it in *them*: and he proposes, with the permission of the Council, to publish his calculations as an appendage to his Second Series of Researches respecting Quaternions, in the Transactions of the Academy. He would only further observe, on the present occasion, that he has made, in these investigations, a frequent use of expressions of the form $\mathrm{Q} + \sqrt{(-1)}\mathrm{Q}'$, where $\sqrt{(-1)}$ is the *ordinary imaginary* of the older algebra, while Q and Q′ are *two different quaternions*, of the kind introduced by him into analysis in 1843, involving the *three new imaginaries* i, j, k, for which the fundamental formula,

$$i^2 = j^2 = k^2 = ijk = -1,$$

holds good. (See the Proceedings of November 13th, 1843.)

And Sir W. R. Hamilton thinks that the name "BIQUATERNION," which he has been for a considerable time accustomed to apply, in his own researches, to an expression of this form $\mathrm{Q} + \sqrt{(-1)}\mathrm{Q}'$, is a designation more appropriate to such expressions than to the entirely different (but very interesting) octonomials of Messrs. J. T. Graves and Arthur Cayley, to which *Octaves* the Rev. Mr. Kirkman, in his paper on *Pluquaternions*, has suggested (though with all courtesy towards the present author), that the name of *biquaternion* might be applied.